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$\sqrt{2} = 1,41421$

$\sqrt{3} = 1,73205$

# A TREATISE

ON

# ELEMENTARY TRIGONOMETRY.

# A TREATISE

ON

# ELEMENTARY TRIGONOMETRY

BY THE

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FELLOW OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE;
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# FROM THE PREFACE TO THE FIRST EDITION.

THE present Work on ELEMENTARY TRIGONOMETRY contains that part of the subject which can conveniently be explained without the use of infinite series. It is intended either for class-teaching or for private study. Accordingly the Examples are numerous and for the most part easy. Those which are not original have been selected from the Cambridge and Army Examination Papers of the last few years.

The Miscellaneous Examples are somewhat more difficult, and should in most cases be postponed until the student reads the subject for the second time.

The order in which the chapters are read may be varied at the discretion of the Teacher.

* * * *

J. B. L.

ETON,
*March*, 1882.

IN the SECOND EDITION a short course has been indicated for the use of Students who wish to read the Solution of Triangles as early as possible. Such

Students are advised to omit every article that is marked with an asterisk.

A double asterisk has been placed before those articles which should be omitted by all Students until they are reading the subject for the second time.

An Appendix containing a description of the Vernier, the Theodolite, the Sextant etc. has been added.

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## CHAPTER I.

### ON MEASUREMENT.

1. IT is usual to say that we have measured any concrete quantity, when we have found out *how many* times it contains some familiar quantity of the same kind.

We say for example, that we have measured a line, when we have found out *how many* feet it contains. We say that we have measured a field, when we have found out *how many* acres or how many square yards it contains.

2. To know the measurement of any quantity then, we must have two things. First, we must have a *unit*, or standard of reference, of the *same kind* as the thing measured. Secondly, we must have the *measure*, or the *number of times* the thing measured contains the unit, or standard quantity.

3. Hence, the **measure** of a quantity is the **number**, and the **unit** is the **concrete quantity**, by means of which it is measured.

*Example* 1. A line contains 261 feet. Here the *measure* or number is 261 and the *unit* a foot.

*Example* 2. What is the measure of $2\frac{1}{2}$ miles when a yard is the unit?

$$2\tfrac{1}{2}\text{ miles} = \tfrac{5}{2} \times 1760\text{ yards}, \\ = 4400\text{ yards} = 4400 \times 1\text{ yard},$$

therefore the measure is 4400 when a yard is the unit.

*Example* 3. What is the unit when the measure of a field of 10 acres is 242?

$$10 \text{ acres} = 242 \times \frac{10 \text{ acres}}{242},$$

$$\therefore \text{ the unit is } \frac{10 \text{ acres}}{242} = 200 \text{ square yards.}$$

*Example* 4. If the unit be $a$ yards, what is the measure of $b$ miles?

$$b \text{ miles} = b \times 1760 \text{ yards},$$

$$\therefore\ b \text{ miles} = \frac{b \times 1760}{a} \times a \text{ yards},$$

$$\therefore \text{ the measure required is } \frac{b \times 1760}{a}.$$

## EXAMPLES. I.

(1) What is the measure of 1 mile when a chain of 66 feet is the unit?

(2) What is the measure of an acre when a square whose side is 22 yards is the unit?

(3) What is the measure of a ton when a weight of 10 stone is the unit?

(4) The length of an Atlantic cable is 2300 miles and the length of the cable from England to France is 21 miles. Express the length of the first in terms of the second as unit.

(5) The measure of a certain field is 22 and the unit 1100 square yards: express the area of the field in acres.

(6) Find the measure of $a$ miles when $b$ yards is the unit.

(7) The measure of a certain distance is $a$ when the unit is $c$ feet. Express the distance in yards.

(8) A certain sum of money has for its measures 24, 240, 960 when three different coins are units respectively. If the first coin is half a sovereign, what are the others?

* 4. The measure of a quantity is the number of times which that quantity contains the unit.

We may express the same thing in different ways,

(i) The *measure* of a quantity is the **ratio** of that quantity to the unit.

(ii) The *measure* of a quantity is the **fraction** that the quantity is of the unit.

* 5. This last statement is in the language of Arithmetic, and the word 'fraction' is to include whole numbers, and improper fractions.

* 6. The following are therefore different forms of the same question:

(i) What is the measure of 4 miles when 66 feet is the unit?

(ii) How many times does 4 miles contain 66 feet?

(iii) What is the ratio of 4 miles to 66 feet?

(iv) What fraction of 66 feet is 4 miles?

*Example* (i) What is the measure of $a$ yards when $b$ feet is the unit?

$$a \text{ yards} = 3a \text{ feet} = \frac{3a}{b} \times b \text{ feet},$$

$$\therefore \text{ the measure is } \frac{3a}{b}.$$

(ii) How many times does $a$ yards contain $b$ feet?

$$\text{As in (i), } a \text{ yards} = \frac{3a}{b} \times b \text{ feet.}$$

*Answer.* $\frac{3a}{b}$ times.

(iii) What is the ratio of $a$ yards to $b$ feet?

$$\text{As in (i), } a \text{ yards} = \frac{3a}{b} \times b \text{ feet},$$

$$\therefore \frac{a \text{ yards}}{b \text{ feet}} = \frac{3a}{b},$$

$$\therefore \text{ the required ratio is } \frac{3a}{b}.$$

(iv) What fraction is $a$ yards of $b$ feet?

$$\text{As in (iii), } \frac{a \text{ yards}}{b \text{ feet}} = \frac{3a}{b},$$

$$\therefore \text{ the required fraction is } \frac{3a}{b}.$$

## * EXAMPLES. II.

(1) The ratio of the area of one field to that of another is 20 : 1, and the area of the first is half a square mile. Find the number of square yards in the second.

(2) The ratio of the heights of two persons is 9 : 8, and the height of the second is 5 ft. 4 in. What is the height of the first?

(3) The measure of a field with 3 acres for unit is 65. Find the ratio of the field to an acre.

(4) One field contains a second of $2\frac{1}{2}$ acres, $6\frac{3}{4}$ times.
What is the measure of the first field in terms of the second?
What is the ratio of the first field to the second?
Express the first as the fraction of the second.

(5) A certain weight is 3·125 of a ton.
What is its measure in terms of 4 cwt.?
How many times does it contain 4 cwt.?
What is its ratio to 4 cwt.?
What fraction is it of 4 cwt.?

(6) The ratio of a certain sum of money to 3 guineas is $\frac{22}{7}$.
Find its measure in terms of one pound.
Find the unit when its measure is 22.

(7) What is the measure of $a$ miles when $b$ chains is the unit?
How many times do $a$ miles contain $c$ chains?
What is the ratio of $a$ miles to $d$ chains?
What fraction is $a$ miles of $k$ chains?

(8) What is the unit when the measure of £20 is $\frac{400}{21}$?
£25 contains a certain sum $\frac{500}{21}$ times. What is that sum?
The ratio of £30 to a certain sum is $\frac{600}{21}$. What is that sum?
The fraction which £10 is of a certain sum is $\frac{200}{21}$. What is that sum?

7. It is explained in Arithmetic, in the application of square measure, that the measure of the area of a rectangle is found in terms of a square unit, by multiplying together the measures of the sides in terms of the corresponding linear unit.

*Example.* Find in square feet the measure of a square surface whose side is 12 feet.

The area is 12 × 12 square feet = 144 × 1 square foot,

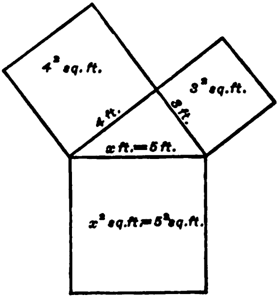
∴ the measure required is 144.

8. We shall apply this result to Euclid I. 47.

*Example* 1. The sides containing the right angle of a right-angled triangle are 3 ft. and 4 ft. respectively; find the length of the hypotenuse.

Let $x$ be the number of feet in the hypotenuse.

Then by Euclid I. 47, the square described on the side of $x$ feet = the sum of the squares described on the sides of 3 feet and 4 feet respectively,

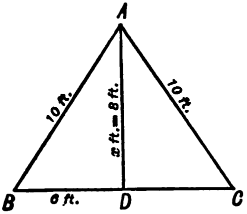


$$\therefore\ x^2 \text{ square feet} = 9 \text{ square feet} + 16 \text{ square feet}$$
$$= 25 \text{ square feet},$$
$$\therefore\ x^2 = 25,$$
$$\therefore\ x = 5.$$

Therefore the length of the hypotenuse is 5 feet.

*Example* 2. Find the length of the perpendicular drawn from the vertex to the base of an isosceles triangle whose equal sides are 10 feet each, and whose base is 12 feet.

Let $ABC$ be the isosceles triangle such that $AB$ is 10 feet, $AC$ is 10 feet and $BC$ is 12 feet.



Draw $AD$ perpendicular to $BC$.

Then because the triangle $ABC$ is isosceles $AD$ will bisect the base $BC$ in $D$; therefore $BD$ is 6 feet.

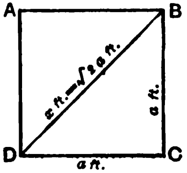
Let $AD$ contain $x$ feet.

Then by Euclid I. 47, the square on $AB$ = the sum of the squares on $BD$ and $AD$.

$$\therefore\ 10^2 \text{ sq. ft.} = 6^2 \text{ sq. ft.} + x^2 \text{ sq. ft.}$$
$$\therefore\ 10^2 = 6^2 + x^2,$$
$$\therefore\ x^2 = 100 - 36,$$
$$\therefore\ x^2 = 64,$$
$$\therefore\ x = 8.$$

Therefore the required length of $AD$ is 8 feet.

*Example* 3. Find the length of the diameter of the square one of whose sides contains $a$ feet.



Let $ABCD$ be the square, so that $AB$ is $a$ feet, and $AD$ is $a$ feet.

Let the diameter $BD$ be $x$ feet.

Then the square on $DB$ = the sum of the squares on $DA$ and $AB$.

$$\therefore\ x^2 \text{ sq. ft.} = a^2 \text{ sq. ft.} + a^2 \text{ sq. ft.}$$

$$\therefore\ x^2 = a^2 + a^2,$$

$$\therefore\ x^2 = 2a^2,$$

$$\therefore\ x = \sqrt{2}\,.\,a.$$

Therefore the required length of the diameter is $\sqrt{2}\,.\,a$ feet.

## EXAMPLES. III.

(1) Find the length of the hypotenuse of a right-angled triangle whose sides are 6 feet and 8 feet respectively.

(2) The hypotenuse of a right-angled triangle is 100 yards and one side is 60 yards: find the length of the other side.

(3) One end of a rope 52 feet long is tied to the top of a pole 48 feet high and the other end is fastened to a peg in the ground. If the pole be vertical and the rope tight, find how far the peg is from the foot of the pole.

(4) The houses in a certain street are 40 feet high and the street 30 feet wide: find the length of the ladder which will reach from the top of one of the houses to the opposite side of the street.

(5) A wall 72 feet high is built at one edge of a moat 54 feet wide; how long must scaling ladders be to reach from the other edge of the moat to the top of the wall?

(6) A field is a quarter of a mile long and three-sixteenths of a mile wide: how many cubic yards of gravel would be required to make a path 2 feet wide to join two opposite corners, the depth of the gravel being 2 inches?

(7) The sides of a rectangular field are $4a$ feet and $3a$ feet respectively. Find the length of its diameter.

(8) If the sides of an isosceles triangle be each $13a$ yards and the base $10a$ yards, what is the length of the perpendicular drawn from the vertex to the base?

(9) Show that the perpendicular drawn from the right angle to the hypotenuse in an isosceles right-angled triangle, each of whose equal sides contains $a$ feet, is $\frac{\sqrt{2}}{2} . a$ ft.

(10) If the hypotenuse of a right-angled isosceles triangle be $a$ yards, what is the length of each side?

(11) Show that the perpendicular drawn from an angular point to the opposite side of an equilateral triangle, each of whose sides contains $a$ feet, is $\frac{\sqrt{3}}{2} . a$ ft.

(12) If in an equilateral triangle the length of the perpendicular drawn from an angular point to the opposite side be $a$ feet, what is the length of the side of the triangle?

(13) Find the ratio of the side of a square inscribed in a circle to the diameter of the circle.

(14) Find the distance from the centre of a circle of radius 10 feet, of a chord whose length is 8 feet.

(15) Find the length of a chord of a circle of radius $a$ yards, which is distant $b$ feet from the centre.

(16) The three sides of a right-angled triangle, whose hypotenuse contains $5a$ feet, are in arithmetical progression: prove that the other two sides contain $4a$ feet and $3a$ feet respectively.

## * CHAPTER II.

### ON INCOMMENSURABLE QUANTITIES.

9. Two numbers are said to be **commensurable** when their ratio can be expressed as an *arithmetical* fraction: that is, as a fraction whose numerator and denominator are both whole numbers.

*Example.* $4{\cdot}9\dot{3}$ and $81\frac{3}{4}$ are two commensurable numbers.

Their ratio is $4\frac{84}{90} \div 81\frac{3}{4} = \frac{444 \times 4}{90 \times 327}$.

10. Two numbers are said to be **incommensurable** when their ratio *cannot* be expressed as an arithmetical fraction.

*Example.* $\sqrt{2}$ and 1 are two incommensurable numbers. For $\frac{\sqrt{2}}{1}$ cannot be expressed exactly as an arithmetical quantity. So $\sqrt{3}$ and $\sqrt{2}$ are two incommensurable numbers.

11. One number alone is said to be *an incommensurable number* when it is incommensurable with *unity*. So that an incommensurable number cannot be expressed as an arithmetical fraction.

*Example.* $\sqrt{2}$ and $\sqrt{3}$, and all surd numbers, are incommensurable numbers.

12. Two *quantities* are said to be commensurable when their *measures* referred to a common unit are commensurable.

*Example.* A mile and a thousand yards are two commensurable quantities. Their measures with a yard for unit are 1760 and 1000; and these are commensurable numbers.

13. Two quantities are said to be incommensurable when their measures referred to a common unit are incommensurable.

*Example.* The side of a square and its diameter are two incommensurable quantities. For if the side of a square contain $a$ feet, the diameter (see Example 3, p. 7) contains $\sqrt{2}\,.\,a$ feet, and therefore the ratio of their measures is $1 : \sqrt{2}$. So that their measures are incommensurable.

14. There is *no practical difficulty* in dealing with incommensurable quantities. We can always find for their measures arithmetical expressions sufficiently accurate for all practical purposes.

15. A little consideration will convince the student that no measurement can in *practice* be made with *absolute* accuracy.

For example: A skilful mechanic is probably satisfied if in measuring some material two or three feet in length the error in his measurement is less than the thirty-second part of an inch. (The thirty-second part of an inch is less than half the height of the smallest letter on this page.) That is to say, he is satisfied if he make no greater error than about a *thousandth part of the whole length* to be measured. He would record such a measurement thus,

$$2 \text{ ft. } 3\tfrac{27}{32} \text{ inches,}$$

which is 891 thirty-second parts of an inch.

16. We will suppose that the length thus measured is the side of a square, and that the workman wishes to know to the same degree of accuracy as his measurement, what is the length of the diameter of the square.

He can find it thus.

The diameter of a square $= \sqrt{2} \times$ its side,

$\therefore$ the diameter of this square $= \sqrt{2} \times 891$ thirty-second parts of an inch.

$$\text{Also } \sqrt{2} = 1{\cdot}414 \text{ nearly,}$$

$$\therefore \sqrt{2} \times 891 = 1{\cdot}414 \times 891$$

$$= 1259{\cdot}8.$$

$\therefore$ the required diameter $= 1260$ thirty-second parts of an inch, nearly.

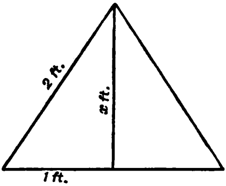
The error being less than one thirty-second part of an inch.

17. The student will be able to see from the above example, that if the value of an incommensurable number is found to 4 figures, a very considerable degree of accuracy is attained. Also that a much greater degree of accuracy is attained for every additional figure.

18. It is no advantage in calculations such as the above to have the value of such quantities as $\sqrt{2}$ calculated to any greater degree of accuracy than the observed measurement. For instance, our calculation being correct as far as the whole numbers are concerned we should gain nothing by using 1·4142 instead of 1·414 for $\sqrt{2}$. This would give us the answer 1260·0 instead of 1259·8; the difference being a fifth of a thirty-second part of an inch, a quantity by hypothesis too small to be of any importance.

*Example* 1. The side of an equilateral triangle contains 2 feet; find, correct to the ten-thousandth part of a foot, the length of the perpendicular drawn from an angular point to the opposite side.

Here (as in Example 2, p. 6), let the perpendicular contain $x$ feet, then



$$x^2 \text{ sq. ft.} = 2^2 \text{ sq. ft.} - 1^2 \text{ sq. ft.},$$

$$\therefore\ x^2 = 4 - 1 = 3,$$

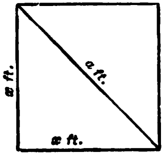
$$\therefore\ x = \sqrt{3},$$

$$= 1{\cdot}7320 \text{ \&c.};$$

$\therefore$ the length of the perpendicular $= 1{\cdot}7320$ feet.

*Example* 2. Find the length, correct to the ten-thousandth part of a foot, of the side of the square described upon a diameter whose length is $a$ feet.

Let $x$ be the number of feet in each of the sides of the square.



Then $x^2$ sq. ft. $+ x^2$ sq. ft. $= a^2$ sq. ft.

$$\therefore x^2 + x^2 = a^2,$$

$$\therefore 2x^2 = a^2,$$

$$\therefore \sqrt{2} . x = a,$$

$$\therefore x = \frac{a}{\sqrt{2}} = \frac{a\sqrt{2}}{2}$$

$$= \frac{a}{2} \times (1{\cdot}4142) = a \times {\cdot}7071\ldots,$$

$\therefore$ the length of the perpendicular $= a \times {\cdot}7071\ldots$feet.

## * EXAMPLES. IV.

(1) Find, correct to the thousandth part of a foot, the length of the diameter of a square whose side is 7 feet.

(2) Find, correct to a yard, the length of the diameter of a square whose side is one mile.

(3) Find, correct to the hundredth part of an inch, the hypotenuse of a right-angled triangle whose sides are 3 ft. 6½ in. and 3 ft. 4 in. respectively.

(4) Find, to the nearest inch, the side of a square whose area is 1000 square yards.

(5) Find, correct to the tenth part of a foot, the diameter of a square field whose area is ten acres.

(6) Find, to the nearest inch, the side of a square field whose area is one acre.

(7) Find the height of an equilateral triangle whose side is 10 feet.

(8) Find the height of an equilateral triangle whose side is 5·32 feet.

(9) The top of a table measures 24·6 inches and 41·3 inches along two adjacent sides. What should the diameter measure if the table is rectangular?

(10) Find, to the nearest inch, the diameter of a lawn-tennis court whose length is 78 feet, and breadth 36 feet, supposing that it is properly marked out.

## CHAPTER III.

### ON THE RELATION BETWEEN THE CIRCUMFERENCE OF A CIRCLE AND ITS DIAMETER.

19. THE circumference of a circle is a line, and therefore it has length.

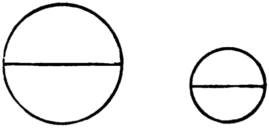
We might imagine the circumference of a circle to consist of a flexible wire; if the circular wire were cut at one point and straightened, we should have a straight line of the same length as the circumference of the circle.

20. A *polygon* is a figure enclosed by any number of straight lines.

21. A *regular* polygon has all its sides equal and all its angles equal.

22. The *perimeter* of a polygon is the sum of its sides.

23. If we have two circles in which the diameter of the first is greater than the diameter of the second, it is evident that the circumference of the first will be greater than the circumference of the second.



24. It seems, therefore, not unlikely that, if the diameter of the first circle be twice that of the second, the circumference of the first will be twice that of the second.

25. And also not unlikely that whatever be the ratio of the circumference of the first circle to its diameter, the same will be the ratio of the circumference of the second circle to its diameter.

This suggests that it is not unlikely that the circumference of a circle $= k$ times its diameter, where $k$ is some *number* which is the same for all circles.

We shall presently prove that this is the case.

26. But although we can prove that

$$\frac{\text{the circumference of a circle}}{\text{its diameter}} = \text{a fixed numerical quantity,}$$

the method of calculating the value of this number is beyond the limits of an elementary treatise.

27. We shall therefore simply state here, what is proved in the Higher Trigonometry,

(i) that this numerical value is incommensurable,

(ii) that it is approximately 3·14159265 &c.

28. When we say that this number is incommensurable we mean (cf. Chapter II.) that its exact value cannot be stated as an *arithmetical* fraction.

It also happens that we have no short *algebraical* expression such as a surd, or combination of surds, which represents it exactly.

So that we have no *numerical* expression whatever, arithmetical nor algebraical, to represent *exactly* the ratio of the circumference of a circle to its diameter.

Hence the universal custom has arisen, of denoting its *exact* value by the letter $\pi$.

29. Thus $\pi$ stands *always* for **the exact value of a certain incommensurable number,** whose approximate value is **3·14159265,** which number is **the ratio of the circumference of any circle to its diameter.**

It cannot be too carefully impressed on the student's memory that $\pi$ stands for this number 3·14159265...&c., and for nothing else; just as 180 stands for the number one hundred and eighty, and for nothing else.

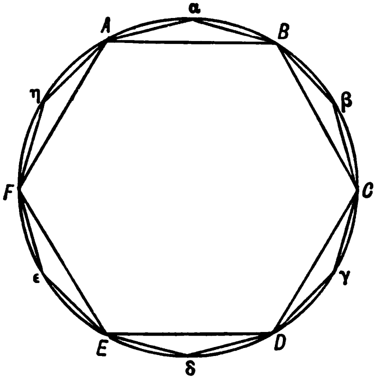
*30. We proceed to prove that the ratio of the circumference of a circle to its diameter is the same for all circles.

The proof depends on the following important principle;

*The length of the circumference of a circle is that to which the length of the perimeter of a regular inscribed polygon approaches, as the number of its sides is continually increased.*

*31. We take for granted that the *straight* line is the *shortest* line that can join two points.

*32. Let *ABCDEF* be any regular polygon inscribed in a circle.



Then the side $AB$ is shorter than any other line joining $AB$, so that the side $AB$ is less than the arc $AB$;

therefore the perimeter of the polygon, viz.

$$AB + BC + CD + DE + EF + FA$$

is less than the sum of the arcs, that is, is less than the circumference of the circle.

Now let each of the *arcs* $AB$, $BC$, etc. be bisected in $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$, $\eta$, and let the lines $A\alpha$, $\alpha B$, $B\beta$, $\beta C$, etc. be joined;

then the figure $A\alpha B\beta C\gamma$ etc. is a regular polygon of twice as many sides as the first polygon.

And since the sides $A\alpha + \alpha B$ are together greater than the side $AB$,

it follows that the perimeter of the second polygon, viz.

$$A\alpha + \alpha B + B\beta + \beta C + C\gamma + \gamma D + \text{etc.}$$

is greater than the perimeter of the first.

But the perimeter of the second polygon is less than the circumference of the circle,

because each side is less than the corresponding arc.

Hence the perimeter of the second polygon is *nearer* the circumference, but is less than the circumference.

By bisecting the arcs of the second polygon we should get a third polygon, whose perimeter is nearer the circumference of the circle than the second.

It is clear that by continuing this process, we can get a polygon whose perimeter is as near as we please to the circumference of the circle. Hence,

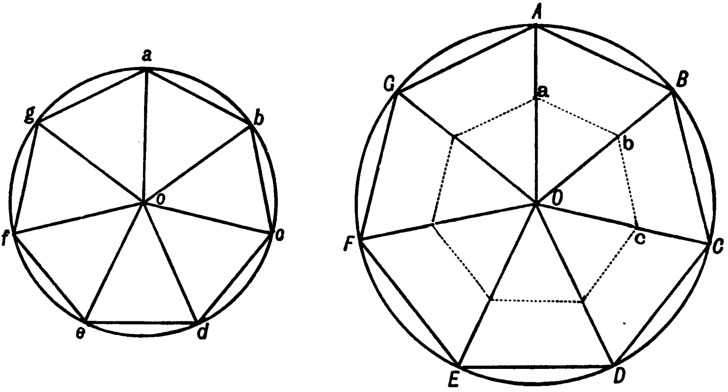
*The length of the circumference of a circle is that to which the length of the perimeter of a regular inscribed polygon approaches, as the number of its sides is continually increased.*

*33. **Prop.** *The ratio of the circumference of a circle to its diameter is the same for all circles.*

Let $ABCDEF$, $abcdef$ be any two circles.

Let a *regular polygon* of any, the same number of sides be inscribed in each of them.

Join $A$, $B$, $C$, etc., $a$, $b$, $c$, etc. the angular points of the polygons to the centres $O$, $o$ respectively.



Then $AOB$ any one of the isosceles triangles in the first figure is equiangular with*, and therefore similar to, $aob$ any one of the isosceles triangles in the second figure.

$$\therefore\ AB : OA = ab : oa, \qquad \text{[Euc. VI. 4.]}$$

and $$BC : OA = bc : oa,$$

and so on.

$$\therefore\ AB + BC + CD + \text{etc.} : OA = ab + bc + cd + \text{etc.} : oa;$$

or, the perimeter of the first polygon is to the radius of the first circle as the perimeter of the second polygon is to the radius of its circle.

---

* For at $O$ and at $o$, four right angles are each divided into the *same* number of equal angles, so that the *vertical* angles $AOB$, $aob$ of the isosceles triangles $AOB$, $aob$ are equal.

This is true *whatever* be the number of the sides of the two polygons.

And therefore it is true *however great* be the number of sides of the two polygons.

But the *circumferences* of the circles are what the *perimeters* of the polygons become, when the number of the sides is indefinitely increased. Therefore the circumference of the first circle is to its radius $OA$ as the circumference of the second circle is to its radius $oa$.

Thus the ratio of the circumference of any circle to its radius is equal to the ratio of the circumference of any other circle to its radius.

So that the ratio $\dfrac{\text{circumference}}{\text{diameter}}$ is the same numerical quantity for all circles. Q. E. D.

34. We said above (Art. 29) that this number is 3·14159 etc., and that it is denoted by $\pi$.

Hence the circumference of any circle of radius $r$

$$= (3\cdot14159265 \text{ etc.}) \times \text{its diameter} = 2\pi r.$$

35. We may notice that $\frac{22}{7} = 3\cdot\dot{1}4285\dot{7}$.

So that $\frac{22}{7}$ and $\pi$ differ by less than a thousandth part of their value.

Also $\frac{355}{113} = 3\cdot1415929$ etc. So that $\frac{355}{113}$ may be used for $\pi$ with sufficient accuracy for any practical purpose.

Hence $\frac{22}{7}$, 3·14159 and $\frac{355}{113}$ are each used for $\pi$ according to the degree of accuracy required.

36. Of these 3·14159 is the most frequently used. The student should notice however that in *dividing* by $\pi$ it will be more convenient to use $\frac{355}{113}$ than 3·14159.

37. The following results are instructive.

The ratio of the perimeter of a regular polygon inscribed in a circle to the diameter of the circle, when the polygon has

four sides, is $2\sqrt{2}$ $=2·8284...$
six sides, is 3 $=3$
eight sides, is $4\sqrt{(2-\sqrt{2})}$ $=3·0614...$
ten sides, is $\frac{5}{2}(\sqrt{5}-1)$ $=3·0901...$
twelve sides, is $3\sqrt{2}(\sqrt{3}-1)$ $=3·1058...$
twenty sides, is $5\{\sqrt{(3+\sqrt{5})}-\sqrt{(5-\sqrt{5})}\}$ $=3·1287...$
sixty sides, is $\frac{15}{2\sqrt{2}}\{(\sqrt{5}-1)(\sqrt{3}+1)$
$-\sqrt{(10+2\sqrt{5})}(\sqrt{3}-1)\}=3·1401...$

These numbers approach 3·14159 as the number of sides is increased, while the surd expression becomes more complicated.

The first of these results the student will be able to verify; the second is proved in Euclid IV. 15. The rest will be proved later on.

*Example* 1. The driving wheel of a locomotive engine is 5 ft. 6 in. high. What is its circumference?

Here we have a circle whose diameter is $5\frac{1}{2}$ feet;

$$\therefore \text{ its circumference} = \pi \times 5·5 \text{ feet},$$
$$= (3·14159...) \times 5·5 \text{ feet},$$
$$= 17·278... \text{ feet}.$$

The circumference is 17 ft. 3 in. approximately.

*Example* 2. If a piece of wire 1 foot long be bent into the form of a circle, what will be the diameter of the circle?

Here the circumference = 1 foot,

that is $\pi \times \text{diameter} = 1$ foot,

$$\therefore \text{ diameter} = \frac{1 \text{ foot}}{\pi} = \tfrac{7}{22} \times 1 \text{ foot}$$

$$= \tfrac{84}{22} \text{ inches}$$

$$= 3{\cdot}8 \text{ inches, nearly.}$$

If a greater degree of accuracy be desired, we must use $\frac{355}{113}$ instead of $\frac{22}{7}$.

We then get, the diameter = 3·7699 inches,
= 3·77 inches very nearly.

## EXAMPLES. V.

In the answers of the first 12 of the following examples $\frac{22}{7}$ is used for $\pi$.

(1) Find the circumference of a circle whose diameter is one yard.

(2) Find the circumference of a circle whose radius is 4 feet.

(3) Find the circumference of a 48 inch bicycle wheel.

(4) The circumference of a circle is 10 feet; find its diameter.

(5) What must be the diameter of a locomotive driving wheel, that it may make 220 revolutions per mile?

(6) How many revolutions does a 36 inch bicycle wheel make per mile?

(7) How many more revolutions per mile does a 50 inch bicycle wheel make than one of 52 inches?

(8) A locomotive whose driving wheel is 5 feet high has an instrument to record the number of revolutions made. What number will the instrument record in running 100 miles?

(9) If the instrument in Question 8 indicates 3 revolutions per second, how many miles per hour is the engine running?

(10) What is the diameter of the driving wheel of a locomotive engine which makes 4 revolutions per second when the engine is going at the rate of 60 miles per hour?

(11) The large hand of the Westminster clock is 11 feet long; how many yards per day does its extremity travel? How far does the extremity move in a minute?

(12) The diameter of the whispering gallery in St Paul's is 108 feet; what is its circumference?

(13) Find the number of inches of wire necessary to construct a figure consisting of a circle with a regular hexagon inscribed in it, one of whose sides is 3 feet.

(14) How many inches of wire would be necessary in a figure similar to that in Question (13), if the circumference of the circle were ten feet?

(15) Find how many inches of wire are necessary to make a figure consisting of a circle and a square inscribed in it, when each side of the square is 2 feet.

(16) How many inches of wire are necessary for a figure similar to that in Question (15), when the circumference of the circle is 12 feet?

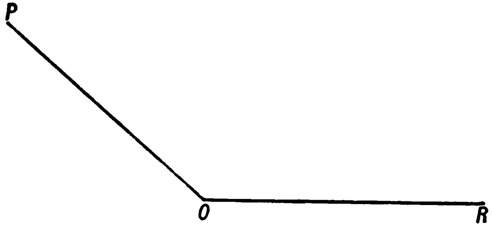
(17) Find the length of string necessary to string the handle of a cricket bat; having given the diameter of the handle $= 1\frac{1}{4}$ in., the length of the handle $= 12$ in., the diameter of the string $= \frac{1}{40}$th of an inch.

# CHAPTER IV.

## ON THE MEASUREMENT OF ANGLES.

38. IN elementary Geometry (Euclid I.—VI.) the angles considered are each always less than two right angles.

For example, in speaking of the angle *ROP* in Euclid we should always mean the angle less than two right angles, not an angle measured in the opposite direction greater than two right angles.

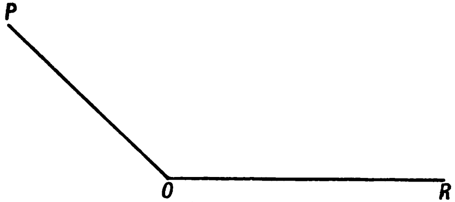


39. In Trigonometry, by *the angle ROP* is meant, not the *present inclination* of the two lines *OR*, *OP* but, the **amount of turning** which *OP* has gone through when, starting from the position *OR*, it has turned about *O* into the position *OP*.

*Example.* Suppose a race run round a circular course. The position of any one of the competitors would be known, if we remark that he has described a certain angle about the centre of the course. Thus, if the distance to be run is three times round, the line joining each competitor to the centre would have to describe an angle of 12 right angles.

When we remark that a competitor has described an angle of $6\frac{2}{3}$ right angles, we record not only his present position, but the total distance he has gone. He would in such a case have gone a little more than one and a half times round the course.

40. Definition. The **angle** between two lines $OR$, $OP$ is the **amount of turning** about the point $O$ which one of the lines $OP$ has gone through in turning from the position $OR$ into the position $OP$.



41. The angle $ROP$ may be the *geometrical* representative of an unlimited number of Trigonometrical angles.

(i) The angle $ROP$ may represent the angle less than two right angles as in Euclid.

In this case $OP$ has turned from the position $OR$ into the position $OP$ by turning about $O$ in the direction *contrary* to that of the hands of a watch.

(ii) The angle $ROP$ may represent the angle described by $OP$ in turning from the position $OR$ into the position $OP$ in the *same* direction as the hands of a watch.

In the first case it is usual to say that the angle $ROP$ is described in the *positive* direction, in the second that the angle is described in the *negative* direction.

(iii) The angle $ROP$ may be the geometrical representation of any of the Trigonometrical angles formed by any number of complete revolutions in the *positive* or in the *negative* direction, added to either of the first two angles. (We shall return to this subject in Chapter IX.)

## EXAMPLES. VI.

Give a geometrical representation of each of the following angles, the starting line being drawn in each case from the turning point towards the right.

1. $+3$ right angles.
2. $+5$ right angles.
3. $+4\frac{1}{2}$ right angles.
4. $+7\frac{1}{4}$ right angles.
5. $-1$ right angle.
6. $10\frac{2}{3}$ right angles.
7. $-10\frac{1}{3}$ right angles.
8. $+4$ right angles.
9. $-4$ right angles.
10. $4n$ right angles.
11. $(4n+2)$ right angles.
12. $-(4n+\frac{1}{2})$ right angles.

42. There are two methods of measuring angles.

(i) The rectangular measure.

(ii) The circular measure.

### RECTANGULAR MEASURE.

43. Angles are always measured *in practice* with the **right angle** (or part of the right angle) as unit.

44. The reasons why the right angle is chosen for a unit are:

(i) All right angles are equal to one another.

(ii) A right angle is practically easy to draw.

(iii) It is an angle whose size is very familiar.

45. The right angle is a large angle, and it is therefore subdivided for practical purposes.

It is usual to explain two methods of subdivision.

(i) The sexagesimal method.

(ii) The centesimal, or decimal, method *.

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* See Art. 55.

## I. *The Sexagesimal Method.*

46. In this method the right angle is divided into 90 equal parts, each of which is called a **degree**; each degree is subdivided into 60 equal parts, each of which is called a **minute**; and each minute is again subdivided into 60 equal parts, each of which is called a **second**.

Instruments used for measuring angles are subdivided accordingly; and the size of an angle is known when, with such an instrument, it has been observed that the angle contains a certain number of degrees, and a certain number of minutes beyond the number of complete degrees, and a certain number of seconds beyond the number of complete minutes.

Thus an angle might be recorded as containing 79 degrees + 18 minutes + 36·4 seconds.

Degrees, minutes, and seconds are indicated respectively by the symbols °, ′, ″, and the above angle would be written

$$79^\circ . 18' . 36{\cdot}4''.$$

## *II. *The Centesimal or Decimal Method.*

47. The other method of subdivision is the Centesimal or Decimal. Here each right angle is divided into 100 equal parts each of which is called a **grade**; each grade is subdivided into 100 equal parts, each of which is called a **minute**; and each minute is again subdivided into 100 equal parts, each of which is called a **second**.

Instruments of observation would be subdivided accordingly, and any observed angle would be recorded as containing so many grades + so many minutes + so many seconds.

Grades, minutes and seconds are indicated respectively by the symbols $^{g}$, $^{\backprime}$, $^{\backprime\backprime}$. So that an angle of 26 grades + 19 minutes + 34·2 seconds would be written

$$36^{g}\ 19^{\backprime}\ 34{\cdot}2^{\backprime\backprime}.$$

*48. It will be observed that this method is simply that of the decimal system of notation.

The above angle for example $= \frac{36}{100} + \frac{19}{10000} + \frac{34 \cdot 2}{1000000}$ of a right angle.

That is ·3619342 of a right angle.
This is equal to 36·19342 of a grade.
Also to 3619·342 of a minute.
Also to 361934·2 of a second.

*Example.* Express $302^{g}\ 2^{\backprime}\ 4{\cdot}6^{\backprime\backprime}$ as the decimal of a right angle.

$$\text{This angle} = \frac{302}{100} + \frac{2}{10000} + \frac{4 \cdot 6}{1000000} \text{ of a right angle}$$
$$= 3{\cdot}0202046 \text{ of a right angle.}$$

*49. Hence, to express an angle given in grades, minutes and seconds as the decimal of a right angle, we have only to observe that the *first and second* decimal places are occupied by the grades, the *third and fourth* decimal places are occupied by the minutes, and the *fifth and sixth* decimal places are occupied by the seconds.

*50. The same observation will enable us to express in grades, minutes, and seconds an angle given as the decimal of a right angle.

*Example.* Express 3·4650023 of a right angle in grades, minutes, and seconds.

3 right angles = 300 grades.
·46 of a right angle = 46 grades.
·00,50 of a right angle = 50 minutes.
·00,00,02,3 of a right angle = 2·3 seconds.

Therefore the angle is $346^{g}\ 50^{\backprime}\ 2{\cdot}3^{\backprime\backprime}$.

## * EXAMPLES. VII.

Express as the decimal of a right angle,

(1) $63^{g}\ 21^{`}\ 18^{``}$.

(2) $104^{g}\ 26^{`}\ 99{\cdot}1^{``}$.

(3) $2^{g}\ 18^{`}\ 27^{``}$.

(4) $3^{g}\ 29^{`}\ 48{\cdot}94^{``}$.

(5) $62^{`}\ 41^{``}$.

(6) $1000^{g}\ 8^{`}\ 12^{``}$.

(7) $32^{g}\ 4^{`}\ 5{\cdot}2^{``}$

(8) $1^{g}\ 2^{`}\ 3{\cdot}4^{``}$

(9) $69^{g}\ 0^{`}\ 7{\cdot}1^{``}$.

(10) $119^{g}\ 3^{`}\ 0{\cdot}45^{``}$.

(11) $1006^{g}\ 18^{`}\ 1^{``}$

(12) $2^{g}\ 26^{`}\ 4{\cdot}8^{``}$.

Express in grades, minutes and seconds,

(13) ·367891 of a right angle.

(14) 1·043021 of a right angle.

(15) ·012003 of a right angle.

(16) ·00102 of a right angle.

(17) ·0625 of a right angle.

(18) 3·02125 of a right angle.

(19) 1·001 of a right angle.

(20) ·0101001 of a right angle.

(21) 6·451 of a right angle.

(22) ·023 of a right angle.

(23) ·00011 of a right angle.

(24) ·00001 of a right angle.

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51. An angle given in degrees, minutes, and seconds may be expressed as the decimal of a right angle by the usual method.

*Example.* Express $39^{0}\ 4'\ 27''$ as the decimal of a right angle.

$$\begin{array}{r|l} 60 & 27 \text{ seconds} \\ \hline 60 & 4{\cdot}45 \text{ minutes} \\ \hline 90 & 39{\cdot}07416666 \text{ etc. degrees} \\ \hline & {\cdot}43415740740 \text{ etc. right angles} \end{array}$$

***Answer.*** $\cdot 43415\dot{7}4\dot{0}$ of a right angle.

52. An angle given as the decimal of a right angle may be expressed in degrees, minutes, and seconds by the converse of the above.

*Example.* Express $\cdot 43415\dot{7}4\dot{0}$ of a right angle in degrees, minutes, and seconds.

·43415740740 etc. right angles
90
39·07416666600 degrees

The last two figures would be 66 if we were to write down the recurring part to more figures.

This gives 39·07416666666 etc. degrees
60
4·4499999960 minutes
that is $4\cdot 44\dot{9}$ minutes
or 4·45 minutes
60
27·00 seconds.

The result is $39^0\ 4'\ 27''$.

*53. We have seen that an angle expressed as the decimal of a right angle can be at once expressed in grades, minutes, and seconds.

Hence an angle expressed in degrees, minutes, and seconds, can be expressed in grades, etc. by first reducing the angle to the decimal of a right angle.

*Example.* Express $39^0\ 4'\ 27''$ in grades, minutes, and seconds.

This angle is $\cdot 43415\dot{7}4\dot{0}$ of a right angle, by Art. 51
and this $=43^{g}\ 41^{\backprime}\ 57\cdot\dot{4}0\dot{7}^{\backprime\backprime}$.

*54. An angle given in grades, minutes, and seconds can be expressed in degrees, minutes, and seconds by first expressing the angle as the decimal of a right angle.

*Example.* Express $\cdot 43^{g}\ 41^{\backprime}\ 57\cdot 40\dot{7}^{\backprime\backprime}$ in degrees, minutes, and seconds.

This angle is $\cdot 43415\dot{7}4\dot{0}$ of a right angle, which is $39^{0}\ 4'\ 27''$ from Art. 52.

## * EXAMPLES. VIII.

Express each of the following angles (i) as the decimal of a right angle, (ii) in grades, minutes, and seconds;

(1) $8^{0}\ 15'\ 27''$.

(2) $6^{0}\ 4'\ 30''$.

(3) $97^{0}\ 5'\ 15''$.

(4) $16^{0}\ 14'\ 19''$.

(5) $132^{0}\ 6'$.

(6) $49^{0}$.

Express in degrees, minutes and seconds,

(7) $1^{g}\ 37^{\backprime}\ 50^{\backprime\backprime}$.

(8) $8^{g}\ 75^{\backprime}$.

(9) $170^{g}\ 45^{\backprime}\ 35^{\backprime\backprime}$.

(10) $24^{g}\ 0^{\backprime}\ 25^{\backprime\backprime}$.

(11) $18^{g}\ 1^{\backprime}\ 15^{\backprime\backprime}$.

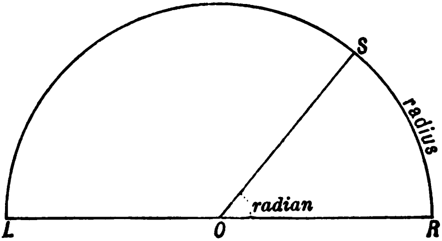
(12) $35^{g}$.

55. The decimal or centesimal system of subdividing a right angle was proposed by the French at the commencement of the present century; but, although it possesses many advantages over the established method, no one has been found willing to undertake the great expense that would have to be incurred in rearranging all tables and all books of reference, and all the records of observations, which would have to be transferred from the old system to the new, before the advantages of the decimal system could be felt. Thus the decimal system of angular measurement has never been used even in France, and in all probability never will be used in practical work.

## ON CIRCULAR MEASURE.

56. By the following construction we get an angle of great importance in Trigonometry.

On the circumference of a circle whose centre is $O$



let an *arc* $RS$ be measured so that its length is equal to the *radius* of the circle, and let $R$ and $S$ be joined to the centre.

57. We are about to prove (Art. 60) that this angle $ROS$ is a *fixed* fraction of a right angle, so that all such angles are equal to one another.

58. We may state the same thing thus—We are about to prove that if we take any number of different circles, and measure on the circumference of each an arc equal in length to its radius, then the angles at the centres of these circles which stand on these arcs respectively, will be all of the same size.

59. **Definition.** The angle which at the centre of a circle stands on an arc equal in length to the radius of the circle is called a **Radian.**

60. *To prove that all Radians are equal to one another.*

Since the Radian at the centre of a circle stands on an arc equal in length to the radius,

and an angle of two right angles at the centre of a circle stands on half the circumference,

Cir. Meas. of an angle is [illegible] answer to the question [illegible] How many times does the arc contain the [illegible]

arc
rad.

and since angles at the centre of a circle are to one another as the arcs on which they stand (Euc. VI. 33),

therefore the radian is to an angle of two right angles as the radius is to half the circumference;
that is, as the diameter is to the whole circumference;
that is, in the constant ratio $1 : \pi$.

$$\text{Therefore the radian} = \frac{2 \text{ right angles}}{\pi}.$$

That is, the radian is a fixed fraction of a right angle.
But all right angles are equal to one another.
Therefore all radians are equal to one another. Q. E. D.

61. Thus the radian possesses the qualification most essential in a unit, viz. it is always the same.

The student will find, in the theoretical part of Trigonometry, that many expressions can be written more shortly when a radian is used for the unit of angle, than when any other unit is used.

62. Thus the reasons why a radian is used as a unit are :

(i) All radians are equal to one another.

(ii) Its use simplifies many formulæ in Theoretical Trigonometry.

63. The system of angular measurement in which a radian is the unit is called **Circular Measure.**

Therefore the *circular measure of an angle* is the *number of radians* which the angle contains.

64. A radian $= \frac{1}{\pi} \times 2$ right angles,

$= \frac{113}{355}$ of $180°$ nearly,

$= 57{\cdot}2957 \ldots$ degrees.

The student should notice that a radian is a little less than an angle of an equilateral triangle.

65. Circular measure is, as we have said, used in *theoretical* investigations, in which the angle under consideration is almost always expressed by a *letter*. This is usually one of the *Greek* letters $\alpha, \beta, \gamma \ldots, \phi, \theta, \psi \ldots$

Strictly speaking these letters represent *numbers*, i.e. measures; so that some unit of *angle* must be *understood* in such an expression as 'the angle $\theta$.' (Art. 2.)

For this reason, when an angle is denoted by a *Greek* letter such as $\alpha$, $\beta$, $\gamma$, etc., $\theta$, $\phi$, $\psi$, etc., it is understood that *circular measure* is the measure used, unless the contrary is *expressly stated.*

So that 'the angle $\theta$' means '$\theta$ radians.'

Similarly 'the angle $\pi$' means '$\pi$ radians' or '3·14159... radians,' that is two right angles.

[*Note.* It will also be convenient, in using such letters as $A$, $B$, $C \ldots S$, $T$, etc. to represent angles, to agree that the *unit* understood with this kind of letter shall be a *degree*, so that when $A$ stands for an angle, that angle contains $A$ degrees.]

66. In *numerical* examples it will be necessary to use some letter ($c$ suppose) to denote a radian.

Thus $2^c$ denotes 'two radians.'

67. Strictly speaking then 'the angle $\theta$' should be written $\theta^c$. (Just as in speaking of 'the angle ninety,' we ought to say ninety *degrees.*) But if it is clearly understood that 'the angle $\theta$' means $\theta$ *radians*, there can be no ambiguity in the expression.

68. The student cannot too carefully notice, that unless an *angle* is obviously referred to, the letters $\theta$, $\phi$,... $\alpha$, $\beta$, .. stand for *mere numbers*.

Thus as we have said above (29) $\pi$ *stands for a number and a number only*, viz. 3·14159......, but in the expression 'the angle $\pi$' that is 'the angle 3·14159......' there must be *some unit* understood. The *unit understood* here is a radian, and therefore 'the angle $\pi$' stands for 3·14159......$^c$, that is *two right angles.*

Hence, *when an angle is referred to,* $\pi$ *is a very convenient abbreviation for* **two right angles.**

69. To express in degrees or grades an angle given in radians, we first express the angle in right angles, remembering that

$$\mathbf{2 \text{ right angles} = \pi \text{ radians.}}$$

*Example.* How many degrees are there in the angle whose circular measure is 2?

$$\text{This angle} = 2 \text{ radians } = 2 \times \frac{2 \text{ right angles}}{\pi} = \frac{4}{\pi} \text{ right angles,}$$

$$= \frac{4 \times 90^0}{\pi} = \frac{360^0}{\pi},$$

$$\therefore \text{ the angle contains } \frac{360}{\pi} \text{ degrees.}$$

*70. If $D$, $G$ and $\alpha$ be the number of degrees, grades and radians respectively in any angle, then

$$\frac{D}{180} = \frac{G}{200} = \frac{\alpha}{\pi}.$$

For each fraction is the ratio of the angle to two right angles.

*Example.* Find the number of degrees in two radians.

Let $D$ be the number, then

$$\frac{D}{180}=\frac{2}{\pi},$$

$$\therefore\ D=\frac{360}{\pi}.$$

## EXAMPLES. IX.

1. Express the following angles in rectangular measure.

(1) $\pi$. (2) $\frac{3\pi}{4}$. (3) $1^c$.
(4) $3^c$. (5) $3{\cdot}14159265^c$ etc. (6) $\frac{2^c}{\pi}$.
(7) $\theta$. (8) $\cdot00314159^c$ etc. (9) $10\pi$.

2. Express the following angles in circular measure.

(1) $180^0$. (2) $360^0$. (3) $60^0$.
(4) $22\frac{1}{2}^0$. (5) $1^0$. (6) $57{\cdot}295^0$ etc.
(7) $n^0$. (8) $\frac{90^0}{\pi}$. (9) $A$.

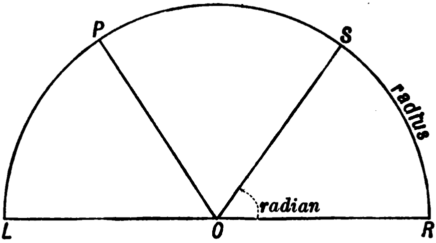
*3. Express the following angles in circular measure.

(1) $33^g\ 33^{\backprime}\ 33{\cdot}\dot{3}^{\backprime\backprime}$. (2) $50^g$. (3) $16{\cdot}\dot{6}^g$.
(4) $1^g$. (5) $1^{\backprime}$. (6) $10^{\backprime\backprime}$.
(7) $n^g$. (8) $\frac{200^g}{\pi}$. (9) $1000^g$.

*4. Find the ratio of

(1) $45^0$ to $\frac{3\pi}{4}$. (2) $60^0$ to $60^g$.
(3) $25^g$ to $22^0\,30'$. (4) $24^g$ to $2^c$.
(5) $1{\cdot}75^c$ to $\frac{100^0}{\pi}$. (6) $1^0$ to $1^c$.

*71. *To prove that the measure of an angle at the centre of a circle in radians* (i.e. *in Circular Measure*) *is the ratio of the arc on which it stands to the radius of the circle.*



Since angles at the centre of a circle are to one another as the arcs on which they stand (Euc. VI. 33),

Therefore any angle $ROP$ at the centre of a circle is to the radian as its arc $RP$ is to the arc of the radian.

But the arc of the radian is equal to the radius,

Therefore $\dfrac{\text{any angle } ROP}{\text{the radian}}$ is equal to $\dfrac{\text{its arc } RP}{\text{the radius}}$.

And therefore any angle $ROP$ is equal to $\dfrac{\text{its arc}}{\text{radius}} \times$ (a radian).

That is, the *measure* $a$ of an angle *in radians* is the ratio $\dfrac{\text{arc}}{\text{radius}}$; or, $a = \dfrac{\text{arc}}{\text{radius}}$.

Hence (cf. Art. 70) $\dfrac{D}{180} = \dfrac{G}{200} = \dfrac{a}{\pi} = \dfrac{\text{arc}}{\text{radius}} \times \dfrac{1}{\pi}$.

*Example.* Find the number of grades in the angle subtended by an arc 46 ft. 9 in. long, at the centre of a circle whose radius is 25 feet.

The angle stands on an arc of $46\frac{3}{4}$ ft. and the radian, at the centre of the same circle, stands on an arc of 25 feet.

$$\therefore \text{ the angle} = \frac{46\frac{3}{4}}{25} \text{ radians}, \quad = \tfrac{187}{100} \times \frac{2 \text{ right angles}}{\pi},$$

$$= \tfrac{187}{100} \times \frac{200^g}{\pi}, = 119^g \text{ nearly.}$$

## * EXAMPLES. X.

(In the answers $\frac{22}{7}$ is used for $\pi$.)

(1) Find the number of radians in an angle at the centre of a circle of radius 25 feet, which stands on an arc of $37\frac{1}{2}$ feet.

(2) Find the number of degrees in an angle at the centre of a circle of radius 10 feet, which stands on an arc of $5\pi$ feet.

(3) Find the number of right angles in the angle at the centre of a circle of radius $3\frac{3}{11}$ inches, which stands on an arc of 2 feet.

(4) Find the number of French minutes in the angle at the centre of a circle of radius 8 ft. 4 inches, which stands on an arc of 1 inch.

(5) Find the length of the arc subtending an angle of $4\frac{1}{2}$ radians at the centre of a circle whose radius is 25 feet.

(6) Find the length of an arc of eighty degrees on a circle of 4 feet radius.

(7) Find the length of an arc of sixty grades on a circle of ten feet radius.

(8) The angle subtended by the diameter of the Sun at the eye of an observer is $32'$; find approximately the diameter of the Sun if its distance from the observer be 90,000,000 miles.

(9) A railway train is travelling on a curve of half a mile radius at the rate of 20 miles an hour; through what angle has it turned in 10 seconds?

(10) A railway train is travelling on a curve of two-thirds of a mile radius, at the rate of 60 miles an hour; through what angle has it turned in a quarter of a minute?

(11) Find approximately the number of English seconds contained in the angle which subtends an arc one mile in length at the centre of a circle whose radius is 4000 miles.

(12) If the radius of a circle be 4000 miles, find the length of an arc which subtends an angle of $1''$ at the centre of the circle.

(13) If in a circle whose radius is 12 ft. 6 in. an arc whose length is ·6545 of a foot subtends an angle of 3 degrees, what is the ratio of the diameter of a circle to its circumference?

(14) If an arc 1·309 feet long subtend an angle of $7\frac{1}{2}$ degrees at the centre of a circle whose radius is 10 feet, find the ratio of the circumference of a circle to its diameter.

(15) On a circle 80 feet in radius it was found that an angle of $22^{0}\ 30'$ at the centre was subtended by an arc 31 ft. 5 in. in length; hence calculate to four decimal places the numerical value of the ratio of the circumference of a circle to its diameter.

(16) If the diameter of the moon subtend an angle of $30'$, at the eye of an observer, and the diameter of the sun an angle of $32'$, and if the distance of the sun be 375 times the distance of the moon, find the ratio of the diameter of the sun to that of the moon.

(17) Find the number of radians in (i.e. the circular measure of) $10''$ correct to 3 significant figures. (Use $\frac{355}{113}$ for $\pi$.)

(18) Find the radius of a globe such that the distance measured upon its surface between two places in the same meridian, whose latitudes differ by $1^{0}\ 10'$, may be one inch.

(19) Two circles touch the base of an isosceles triangle at its middle point, one having its centre at, and the other passing through the vertex. If the arc of the greater circle included within the triangle be equal to the arc of the lesser circle without the triangle, find the vertical angle of the triangle.

(20) By the construction in Euc. I. 1, prove that the unit of circular measure is less than $60^{0}$

(21) On the 31st December the Sun subtends an angle of 32′ 36″, and on 1st July an angle of 31′ 32″; find the ratio of the distances of the Sun from the observer on those two days.

(22) Show that the measure of the angle at the centre of a circle of radius $r$, which stands on an arc $a$, is $\frac{k \cdot a}{r}$, where $k$ depends solely on the unit of angle employed.

Find $k$ when the unit is (i) a radian, (ii) a degree.

*72. Questions concerning angles expressed in different systems of measurement are easily solved by *expressing each angle in right angles.*

*Example* 1. The sum of the measure of an angle in degrees and twice its measure in radians is $23\frac{2}{7}$, find its measure in degrees ($\pi=\frac{22}{7}$).

Let the angle contain $x$ right angles.

Then the measure of the angle in degrees $=90x$,

„ „ „ „ „ radians $=\frac{\pi}{2}x$.

$$\therefore\ 90x+2\cdot\frac{\pi}{2}x=23\tfrac{2}{7},$$

$$\therefore\ 90x+\tfrac{22}{7}x=\tfrac{163}{7},$$

$$\therefore\ 652x=163,$$

$$\therefore\ x=\tfrac{1}{4}.$$

The angle is $\frac{1}{4}$ of a right angle, that is $22\frac{1}{2}^{\circ}$, nearly.

*Example* 2. The three angles of a triangle are in arithmetical progression, and the measure of the least in grades is to that of the greatest in circular measure as $120 : \pi$. Express each angle in degrees.

Let the angles contain $x-y$, $x$, $x+y$ **right angles** respectively; they are then in A.P.

Their *sum* is $3x$ right angles; but since they are the angles of a triangle, their *sum* is 2 right angles;

$$\therefore 3x=2,$$

$$\therefore x=\tfrac{2}{3}.$$

Again, the least angle contains $(x-y)\times 100$ grades, and the greatest angle contains $(x+y)\frac{\pi}{2}$ radians,

$$\therefore 100(x-y) : \frac{\pi}{2}(x+y)=120 : \pi.$$

$$\therefore 100(x-y)=60(x+y),$$

$$\text{or, } 40x=160y,$$

$$\text{or, } x=4y.$$

$$\therefore 4y=\tfrac{2}{3},$$

$$\text{because } x=\tfrac{2}{3},$$

$$\text{or, } y=\tfrac{1}{6}.$$

$$\therefore x-y=\tfrac{2}{3}-\tfrac{1}{6}=\tfrac{1}{2},\quad x=\tfrac{2}{3},\quad x+y=\tfrac{2}{3}+\tfrac{1}{6}=\tfrac{5}{6}.$$

Thus the angles contain $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{5}{6}$ right angles respectively;

therefore the angles are $45^0$, $60^0$, $75^0$.

## * * EXAMPLES. XI.

(In the following examples the answers will be given in terms of $\pi$.)

(1) The sum of the degrees and of the grades in a certain angle is 38; find its circular measure.

(2) The difference of two angles is $20^g$, and their sum is $48^0$; find them.

(3) One angle is double of a second, and the sum of their measures in degrees and in grades respectively is 140; express the angles in degrees.

(4) Two angles are in the ratio of 4 : 5, and the difference of their measures in grades and in degrees respectively is $2\frac{1}{2}$; find the angles in degrees.

(5) The difference between two angles is $\frac{\pi}{9}$, and their sum is 56 degrees; find the angles.

(6) If the three angles of a triangle are in arithmetical progression, show that the mean angle is $60^0$.

(7) The three angles of a triangle are in arithmetical progression, and the number of grades in the least is to the number of degrees in the mean as 5 : 6. Find the angles in degrees.

(8) The three angles of a triangle are in arithmetical progression, and the number of grades in the greatest is to the number of degrees in the sum of the other two as 10 : 11. Find the angles in degrees.

(9) The three angles of a triangle are in arithmetical progression, and the number of grades in the least is to the number of radians in the greatest as $200 : 3\pi$. Express the angles in grades.

(10) If $D$ be the number of degrees and $G$ the number of grades in any angle, prove that $G - D = \frac{1}{9}D$.

(11) If $M$ be the number of English minutes and $m$ the number of French minutes in any angle, prove that

$$2M - m = \tfrac{4}{27}M.$$

(12) If $G$, $D$ and $C$ be the number of grades, degrees and radians in any angle, prove that $G - D = \dfrac{20C}{\pi}$.

(13) If an angle be expressed in French minutes, show that it will be transferred to English minutes by multiplying by ·54.

(14) Divide $33^0\ 6'$ into two parts so that the number of English seconds in one part may be equal to the number of French seconds in the other part.

(15) Find the ratio of $9^0\ 27'$ to $12^g\ 50^{\backprime}$.

(16) Find the number of radians in an angle of $n$ English minutes.

(17) Express in each of the three systems of angular measurement the angles

(i) of a regular hexagon,
(ii) of a regular octagon,
(iii) of a regular quindecagon.

(18) Show that the number of degrees in an angle of a regular decagon is to the number of grades in an angle of a regular pentagon in the ratio of 6 : 5.

(19) Show that the number of grades in an angle of a regular pentagon is equal to the number of degrees in an angle of a regular hexagon.

(20) Find in English minutes the difference between the angle of a regular polygon of 48 sides and two right angles.

(21) If we take for unit the angle between a side of a regular quindecagon and the next side produced, find the measures (i) of a right angle, (ii) of a radian.

(22) Find the unit when the sum of the measures of a degree and of a grade is 1.

(23) What is the unit when the sum of the measures of $9^0$ and of $5^g$ is $\frac{3}{20}$?

(24) If the measure of $b$ grades is $a$, find the measure of $c$ degrees.

(25) What is the unit when the sum of the measures of $a$ grades and of $b$ degrees is $c$?

(26) The number of grades in a certain angle exceeds the number of degrees in it by $\frac{1}{10}$ of the number of degrees in a radian. If this angle be taken as unit, what is the measure of a right angle?

(27) The three numbers which express the three angles of a triangle are all equal, and the units of angle in each are respectively a degree, a grade and the sum of a degree and a grade; express each of the angles in circular measure.

(28) The three angles of a triangle have the same measure when expressed in degrees, grades and radians respectively; find this measure.

(29) The measures of the angles of a triangle in degrees, grades and radians respectively are in the ratio of 1 : 10 : 100; find the number of radians in the smallest angle.

(30) The interior angles of an irregular polygon are in A. P.; the least angle is $120^0$; and the common difference $5^0$: find the number of sides.

# CHAPTER V.

## THE TRIGONOMETRICAL RATIOS.

√ 73. Let *ROE* be any angle (see the figure in Art. 83). In one of the lines containing the angle take any point *P*, and from *P* draw *PM* perpendicular to the other line *OR*.

Then, in the right-angled triangle *OPM*, formed from the angle *ROE*,

(i) the side *MP*, which is *opposite the angle under consideration*, is called the **perpendicular**;

(ii) the side *OP*, which is *opposite the right angle*, is called the **hypotenuse**;

(iii) the *third* side *OM*, which is adjacent to the right angle and to the angle under consideration, is called the **base**.

From these three,—perpendicular, hypotenuse, base,—we can form *three* different sets containing two each.

The **ratios** or *fractions* formed from these sets, viz.

$$\text{(i) } \frac{\text{perpendicular}}{\text{hypotenuse}}, \quad \text{(ii) } \frac{\text{base}}{\text{hypotenuse}}, \quad \text{(iii) } \frac{\text{perpendicular}}{\text{base}},$$

and the ratios formed by inverting each of them, viz.

$$\text{(iv) } \frac{\text{hypotenuse}}{\text{perpendicular}}, \quad \text{(v) } \frac{\text{hypotenuse}}{\text{base}}, \quad \text{(vi) } \frac{\text{base}}{\text{perpendicular}},$$

will be found to be of great importance in treating of any angle *ROE*. Accordingly to each of these six ratios has been given a separate *name* (Art. 75).

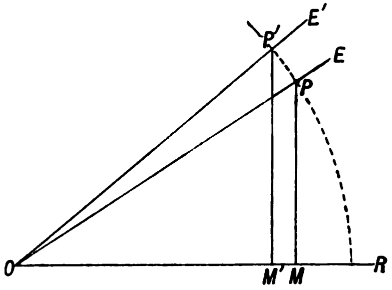
74. The student should observe carefully

(i) that each *ratio*, such as $\frac{\text{perpendicular}}{\text{hypotenuse}}$, is a mere *number;*

(ii) that, as we shall prove in Art. 83, these ratios remain unchanged as long as the angle remains unchanged;

(iii) that if the angle be altered ever so slightly, there is a consequent alteration in the value of these ratios.

[For, let $ROE$, $ROE'$ be two angles which are nearly equal;

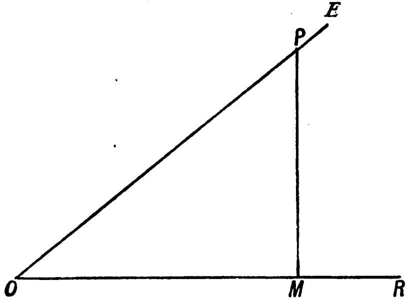


Let $OP = OP'$; then $OM$ is *not*$= OM'$, and therefore the ratios $\frac{OM}{OP}$ and $\frac{OM'}{OP'}$ are *not* equal; also $MP$ is *not*$= M'P'$ and therefore the ratios $\frac{MP}{OP}$ and $\frac{M'P'}{OP'}$ are *not* equal.]

(iv) that by giving names to these ratios we are enabled to apply the methods of Algebra to the Geometry of Euclid VI., just as in Chapter I. we applied the methods of Algebra to Euc. I. 47.

The student is recommended to pay careful attention to the following definitions. He should be able to write them out in the exact words in which they are printed.

✓ 75. **Definition.** *To define the three principal Trigonometrical Ratios of an angle.*



Let $ROE$ be an angle.

In $OE$ one of the lines containing the angle take any point $P$, and from $P$ draw $PM$ perpendicular to the other line $OR$, or, if necessary, to $RO$ produced.

Then, in the right-angled triangle $OPM$, the side $MP$, which is *opposite the angle under consideration*, is called the *perpendicular*.

The side $OP$, which is *opposite the right angle*, is called the *hypotenuse*.

The *third* side $OM$ (which is adjacent to the right angle and to the angle under consideration) is called the *base*.

Then the ratio

(i) $\frac{MP}{OP} = \frac{\text{perpendicular}}{\text{hypotenuse}}$ is called the **sine** of the angle $ROE$.

(ii) $\frac{OM}{OP} = \frac{\text{base}}{\text{hypotenuse}}$ ,, **cosine** ,,

(iii) $\frac{MP}{OM} = \frac{\text{perpendicular}}{\text{base}}$ ,, **tangent** ,,

**Note.** The order of the letters, in $MP$, $OM$ and $OP$, indicates direction and decides their algebraical signs. [Art. 132.]

76. If $A$ stand for the angle $ROE$, these ratios are called sine $A$, cosine $A$ and tangent $A$, and are usually abbreviated thus:

$$\sin A, \qquad \cos A, \qquad \tan A.$$

77. There are three other Trigonometrical Ratios, formed by *inverting* the sine, cosine and tangent respectively, which are called the cosecant, secant, and cotangent respectively.

✓ 78. *To define the three other Trigonometrical Ratios of any angle.*

The same construction and figure as in Art. 75 being made, then the ratio

(iv) $\frac{OP}{MP} = \frac{\text{hypotenuse}}{\text{perpendicular}}$ is called the **cosecant** of the angle $ROE$.

(v) $\frac{OP}{OM} = \frac{\text{hypotenuse}}{\text{base}}$ ,, **secant** ,,

(vi) $\frac{OM}{MP} = \frac{\text{base}}{\text{perpendicular}}$ ,, **cotangent** ,,

79. Thus if $A$ stand as before for the angle $ROE$, these ratios are called cosecant $A$, secant $A$, and cotangent $A$. They are abbreviated thus,

$$\text{cosec}\, A, \qquad \sec A, \qquad \cot A.$$

80. From the definition it is clear that

$$\text{cosec}\, A = \frac{1}{\sin A},$$

$$\sec A = \frac{1}{\cos A}$$

$$\cot A = \frac{1}{\tan A}.$$

81. The above definitions **apply to an angle of any magnitude.** (We shall return to this subject in Chapter X.)

For the present the student may confine his attention to angles which are each less than a right angle.

82. The powers of the Trigonometrical Ratios are expressed as follows:

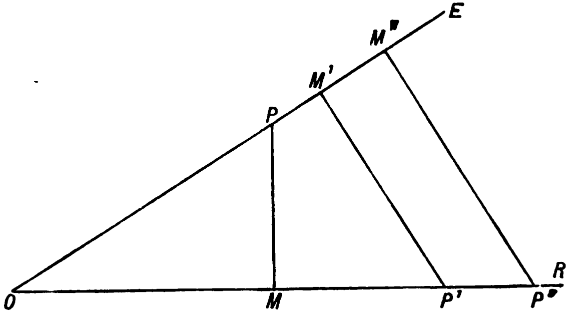
$$(\sin A)^2, \text{ i.e. } \left(\frac{\text{perpendicular}}{\text{hypotenuse}}\right)^2, \text{ is written } \quad \sin^2 A,$$

$$(\cos A)^2, \text{ i.e. } \left(\frac{\text{base}}{\text{hypotenuse}}\right)^2, \text{ is written } \quad \cos^2 A,$$

and so on.

The student must notice that 'sin $A$' is a *single symbol.* It is the *name* of a *number*, or *fraction*, belonging to the angle $A$; and if it be at any time convenient, we may denote *sin* $A$ by a *single letter*, such as $s$ or $x$. Also $\sin^2 A$ is an abbreviation for $(\sin A)^2$, that is for $(\sin A) \times (\sin A)$. Such abbreviations are used because they are *convenient.*

83. *The Trigonometrical Ratios are always the same for the same angle.*



Take any angle $ROE$; let $P$ be any point in $OE$ one of the lines containing the angle, and let $P'$, $P''$ be any two points in $OR$ the other line containing the angle. Draw $PM$ perpendicular to $OR$, and $P'M'$, $P''M''$ perpendiculars to $OE$.

Then the three triangles $OMP$, $OM'P'$, $OM''P''$ each contain a right angle, and they have the angle at $O$ common; therefore their third angles must be equal.

Thus the three triangles are equiangular.

Therefore the ratios $\frac{MP}{OP}$, $\frac{M'P'}{OP'}$, $\frac{M''P''}{OP''}$ are all equal. (Eu. VI. 4.)

But each of these ratios is $\frac{\text{perpendicular}}{\text{hypotenuse}}$ *with reference to the angle at* $O$; that is, they are each *sin* $ROE$.

Thus, sin $ROE$ is the same *whatever* be the position of the point $P$ on *either* of the lines containing the angle $ROE$.

Therefore sin $ROE$ is always the same.

84. A similar proof holds good for each of the other ratios.

85. Also if two angles are equal, it is clear that the numerical values of their Trigonometrical Ratios will be the same.

We have already shown (Art. 74), that the values of these ratios are different for different angles.

Hence for each particular value of $A$, sin $A$, cos $A$, tan $A$, etc. have *definite numerical values.*

*Example.* We shall prove (Art. 92) that

$$\sin 30^0 = \tfrac{1}{2} = \cdot 5,\quad \cos 30^0 = \frac{\sqrt{3}}{2} = \cdot 8660\ldots,\quad \tan 30^0 = \frac{1}{\sqrt{3}} = \cdot 577\ldots$$

86. In the following examples the student should notice

(i) the *angle* referred to,

(ii) that there is a *right angle* in the same triangle as the angle referred to,

(iii) the *perpendicular*, which is opposite the angle referred to, and is perpendicular to one of the lines containing the angle,

(iv) the *hypotenuse*, which is opposite the right angle,

(v) the *base*, the third side of the triangle.

*Example.* In the second figure on the next page, in which $BDA$ is a right angle, find sin $DBA$ and cos $DBA$.

In this case

(i) $DBA$ is the *angle*.

(ii) $BDA$ is a *right angle* in the same triangle as the angle $DBA$.

(iii) $DA$ is the *perpendicular*, for it is opposite $DBA$ and is perpendicular to $BD$.

(iv) $BA$ is the *hypotenuse*.

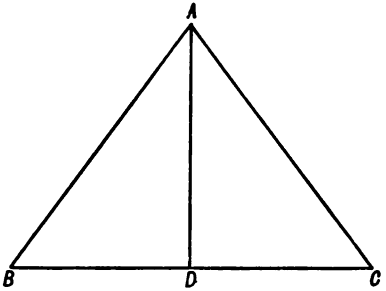
(v) $BD$ is the *base*.

$$\text{Therefore}\quad \sin DBA, \text{ which is } \frac{\text{perpendicular}}{\text{hypotenuse}}, = \frac{DA}{BA},$$

$$\cos DBA, \text{ which is } \frac{\text{base}}{\text{hypotenuse}}, = \frac{BD}{BA}.$$

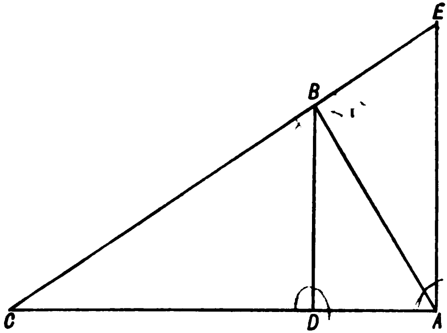
## EXAMPLES. XII.

(1) Let $ABC$ be any triangle and let $AD$ be drawn perpendicular to $BC$. Write down the *perpendicular*, and the *base* when the following angles are referred to: (i) the angle $ABD$, (ii) the angle $BAD$, (iii) the angle $ACD$, (iv) the angle $DAC$.



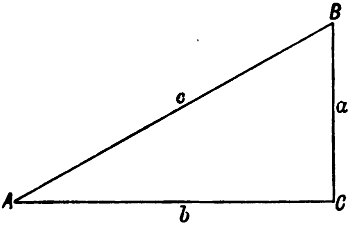
(2) Write down the following ratios in the above figure; (i) sin $BAD$, (ii) cos $ACD$, (iii) tan $DAC$, (iv) sin $ABD$, (v) tan $BAD$, (vi) sin $DAC$, (vii) cos $DCA$, (viii) tan $DCA$, (ix) cos $ABD$, (x) sin $ACD$.

(3) Let $ACB$ be any angle and let $ABC$ and $BDC$ be right angles; (see next figure). Write down *two* values for each of the following ratios; (i) sin $ACB$, (ii) cos $ACB$, (iii) tan $ACB$, (iv) sin $BAC$, (v) cos $BAC$, (vi) tan $BAC$.



(4) In the accompanying figure $BDC$, $CBA$ and $EAC$ are right angles. Write down (i) sin $DBA$, (ii) sin $BEA$, (iii) sin $CBD$, (iv) cos $BAE$, (v) cos $BAD$, (vi) cos $CBD$, (vii) tan $BCD$, (viii) tan $DBA$, (ix) tan $BEA$, (x) tan $CBD$, (xi) sin $DAB$, (xii) sin $BAE$.

(5) Let $ABC$ be a right-angled triangle such that $AB=5$ ft., $BC=3$ ft., then $AC$ will be 4 ft.



Find the sine, cosine and tangent of the angles at $A$ and $B$ respectively.

In the above triangle if $A$ stand for the angle at $A$ and $B$ for the angle at $B$, show that $\sin^2 A+\cos^2 A=1$, and that $\sin^2 B+\cos^2 B=1$.

(6) If $ABC$ be any right-angled triangle with a right angle at $C$, and let $A$, $B$, and $C$ stand for the angles at $A$, $B$ and $C$ respectively, and let $a$, $b$ and $c$ be the measures of the sides opposite the angles $A$, $B$ and $C$ respectively.

Show that $\sin A=\frac{a}{c}$, $\cos A=\frac{b}{c}$, $\tan A=\frac{a}{b}$

Show also that $\sin^2 A+\cos^2 A=1$.

Show also that (i) $a=c.\sin A$, (ii) $b=c.\sin B$, (iii) $a=c.\cos B$, (iv) $b=c.\cos A$, (v) $\sin A=\cos B$, (vi) $\cos A=\sin B$, (vii) $\tan A=\cot B$.

(7) The sides of a right-angled triangle are in the ratio 5 : 12 : 13. Find the sine, cosine and tangent of each acute angle of the triangle.

(8) The sides of a right-angled triangle are in the ratio 1 : 2 : $\sqrt{3}$. Find the sine, cosine and tangent of each acute angle of the triangle.

(9) Prove that if $A$ be either of the angles of the above two triangles $\sin^2 A+\cos^2 A=1$.

# CHAPTER VI.

## ON THE TRIGONOMETRICAL RATIOS OF CERTAIN ANGLES.

87. The Trigonometrical Ratios of an angle are *numerical quantities simply*, as their name ratio implies. They are in nearly all cases incommensurable numbers.

Their practical value has been found for all angles between 0 and 90°, which differ by 1′; and a list of these values will be found in any volume of Mathematical Tables.

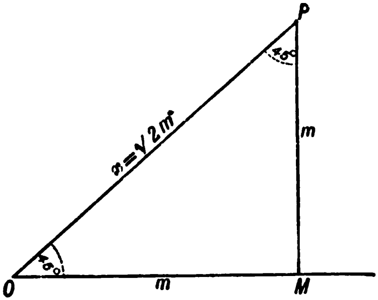
The student is recommended to get a copy of Chambers' Mathematical Tables for instruction and reference.

88. The finding the values of these Ratios has involved a large amount of labour; but, as the results have been published in Tables, the finding the Trigonometrical Ratios does not form any part of a student's work, except to exemplify the method employed.

89. The general method of finding Trigonometrical Ratios belongs to a more advanced part of the subject than the present, but there are certain angles whose Ratios can be found in a simple manner.

90. *To find the sine, cosine and tangent of an angle of* 45°.

If one angle of a right-angled triangle be 45°, that is, the half of a right angle, the third angle must also be 45°. Hence 45° is one angle of an isosceles right-angled triangle.



Let $POM$ be an isosceles triangle such that $PMO$ is a right angle, and $OM = MP$. Then $POM = OPM = 45°$.

Let the measures of $OM$ and of $MP$ each be $m$. Let the measure of $OP$ be $x$.

Then $$x^2 = m^2 + m^2 = 2m^2;$$

$$\therefore\ x = \sqrt{2}\,.\,m.$$

Hence, $$\sin 45° = \sin POM = \frac{MP}{OP} = \frac{m}{\sqrt{2}\,.\,m} = \frac{1}{\sqrt{2}},$$

$$\cos 45° = \cos POM = \frac{OM}{OP} = \frac{m}{\sqrt{2}\,.\,m} = \frac{1}{\sqrt{2}},$$

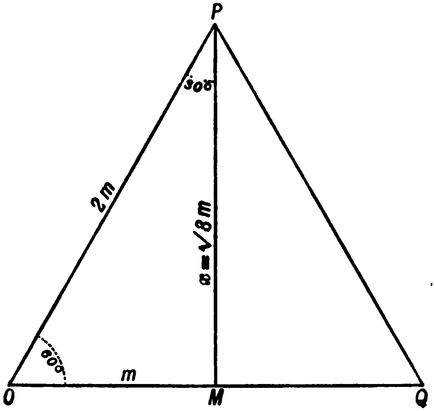
$$\tan 45° = \tan POM = \frac{MP}{OM} = \frac{m}{m} = \frac{1}{1} = 1.$$

91. *To find the sine, cosine and tangent of* 60°.

In an equilateral triangle, each of the equal angles is 60°, because they are each one third of 180°. And if we draw a perpendicular from one of the angular points of the triangle to the opposite side, we get a right-angled triangle in which one angle is 60°.

Let $OPQ$ be an equilateral triangle. Draw $PM$ perpendicular to $OQ$. Then $OQ$ is bisected in $M$.

Let the measure of $OM$ be $m$; then that of $OQ$ is $2m$ and therefore that of $OP$ is $2m$.



Let the measure of $MP$ be $x$.

Then $$x^2 = (2m)^2 - m^2 = 4m^2 - m^2 = 3m^2,$$
$$\therefore\ x = \sqrt{3}\,.\,m.$$

Hence, $$\sin 60^\circ = \sin POM = \frac{MP}{OP} = \frac{\sqrt{3}\,.\,m}{2m} = \frac{\sqrt{3}}{2},$$
$$\cos 60^\circ = \cos POM = \frac{OM}{OP} = \frac{m}{2m} = \frac{1}{2},$$
$$\tan 60^\circ = \tan POM = \frac{MP}{OM} = \frac{\sqrt{3}\,.\,m}{m} = \frac{\sqrt{3}}{1} = \sqrt{3}.$$

92. *To find the sine, cosine and tangent of* 30°.

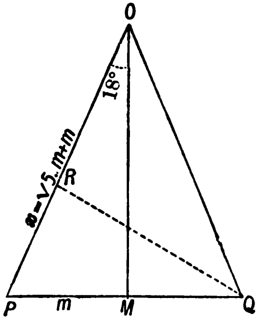
With the same figure and construction as above, we have the angle $OPM = 30^\circ$, since it is a half of $OPQ$, i.e. of 60°.

Hence, $$\sin 30^\circ = \sin OPM = \frac{MO}{PO} = \frac{m}{2m} = \frac{1}{2},$$
$$\cos 30^\circ = \cos OPM = \frac{PM}{PO} = \frac{\sqrt{3}\,.\,m}{2m} = \frac{\sqrt{3}}{2},$$
$$\tan 30^\circ = \tan OPM = \frac{MO}{PM} = \frac{m}{\sqrt{3}\,.\,m} = \frac{1}{\sqrt{3}}.$$

## **93. *To find the sine of* 18°.

(2) 1 1 ut)

In the 10th Prop. of Euclid IV. a triangle is described such that each of the angles at the base is double of the third angle. [See also Example (6) p. 140.]



Let $POQ$ be such a triangle, and let the vertical angle $POQ$ contain $n$ degrees; then

$$n + 2n + 2n = 180,$$

$$n = 36.$$

Draw $OM$ perpendicular to $PQ$, bisecting the angle $POQ$. Then since $QOP = 36°$, $MOP = 18°$. Also $PM = MQ$.

Let the measure of $MP$ be $m$, and the measure of $OP$ be $x$. From $OP$ cut off $OR = PQ$. Then by Euclid IV. 10 $PO \,.\, PR = PQ^2$.

$$\therefore\ x(x - 2m) = (2m)^2,$$

$$\therefore\ x^2 - 2mx + m^2 = 4m^2 + m^2 = 5m^2,$$

$$\therefore\ x - m = \sqrt{5} \,.\, m,$$

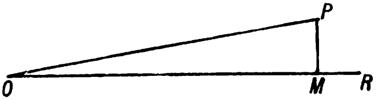
$$\therefore\ x = \sqrt{5} \,.\, m + m,$$

$$\therefore \sin 18° = \sin MOP = \frac{MP}{OP} = \frac{m}{\sqrt{5} \,.\, m + m} = \frac{1}{\sqrt{5} + 1} = \frac{\sqrt{5} - 1}{4}.$$

X

### 94. *To find the sine, cosine and tangent of* 0°.

By this is meant,—*To find the values, if any, to which the Trigonometrical Ratios of a very small angle approach, as the angle is continually diminished.*



Let $ROP$ be a small angle. Draw $PM$ perpendicular to $OR$, and let $OP$ be always of the same length, so that $P$ lies on a circle whose centre is $O$.

Then if the angle $ROP$ be diminished, we can see that $MP$ is diminished also, and that consequently $\frac{MP}{OP}$, which is sin $ROP$, is diminished. And, by diminishing the angle $ROP$ sufficiently, *we can make $MP$ as small as we please*, and therefore *we can make* sin $ROP$ *smaller than any assignable number however small that number may be.*

Thus we see that the value to which sin $ROP$ approaches as the angle is diminished, is 0.

This is expressed by saying, $\sin 0° = 0$.................. i.

Again, as the angle $ROP$ diminishes, $OM$ approaches $OP$ in length; and cos $ROP$, which is $\frac{OM}{OP}$, approaches in value to $\frac{OP}{OP}$, i.e. to 1.

This is expressed by saying, $\cos 0° = 1$ ..................... ii.

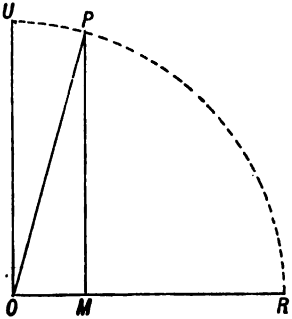
Also, tan $ROP$ is $\frac{MP}{OM}$; and we have seen that $MP$ approaches 0, while $OM$ does not; ∴ tan $ROP$ approaches 0.

This is expressed by saying, $\tan 0° = 0$ ............... iii.

95. *To find the sine, cosine and tangent of* 90°.

By this is meant,—*To find the values, if any, to which the Trigonometrical Ratios of an angle approach, as the angle approaches a right angle.*

Let *ROU* be a right angle = 90°.



Draw *ROP* nearly a right angle; draw *PM* perpendicular to *OR*, and let *OP* be always of the same length, so that *P* lies on a circle whose centre is *O*.

Then, as the angle *ROP* approaches to *ROU*, we can see that *MP* approaches *OP*, while *OM* continually diminishes.

Hence when *ROP* approaches 90°, sin *ROP*, which is $\frac{MP}{OP}$, approaches in value to $\frac{OP}{OP}$, that is to $\frac{1}{1}$, i.e. to 1.

Hence we say, that sin 90° = 1 .......................... i.

Again, when *ROP* approaches 90°, cos *ROP*, which is $\frac{OM}{OP}$, approaches in value to $\frac{0}{OP}$, that is to 0.

Hence we say, that cos 90° = 0 ...................... ii.

Again, when $ROP$ approaches $90^\circ$, tan $ROP$ which is $\frac{MP}{OM}$ approaches in value to $\frac{OP}{\text{a quantity which approaches } 0}$.

But in any fraction whose numerator does not diminish, the smaller the denominator the greater is the value of that fraction; and if the denominator continually diminishes the value of the fraction continually increases.

Hence, *tan $ROP$ can be made larger than any assignable number by making the angle $ROP$ approach* $90^\circ$ *near enough.*

This is what we mean when we say, that

$$\tan 90^\circ \text{ is } \textbf{infinity}, \text{ or, } \tan 90^\circ = \infty \quad \ldots\ldots\ldots\ldots \text{ iii.}$$

96. The following table exhibits the results of this Chapter.

| angle | $0^0$ | $18^0$ | $30^0$ | $45^0$ | $60^0$ | $90^0$ |
|---|---|---|---|---|---|---|
| sine | 0 | $\frac{\sqrt{5}-1}{4}$ | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | 1 |
| cosine | 1 | | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 |
| tangent | 0 | | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | $\infty$ |

The student may notice that the sine increases with the angle, while the cosine diminishes as the angle increases.

Also that the squares of the sines of $0^0$, $30^0$, $45^0$, $60^0$ and $90^0$ are respectively $0$, $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$ and $\frac{4}{4}$, and that the squares of the cosines of the same angles are $\frac{4}{4}$, $\frac{3}{4}$, $\frac{2}{4}$, $\frac{1}{4}$, and 0.

## EXAMPLES. XIII.

If $A=90^0, B=60^0, C=30^0, D=45^0, E=18^0$, prove the following:

(1) $2.\sin D.\cos D=\sin A$. (2) $2.\sin C.\cos C=\sin B$.

(3) $\cos^2 B-\sin^2 B=1-2\sin^2 B$.

(4) $\sin B.\cos C+\sin C.\cos B=\sin A$.

(5) $\cos^2 D-\sin^2 D=\cos A$. (6) $4.\sin^2 E+2.\sin E=1$.

(7) $\sin^2 B+\cos^2 B=1$. (8) $\cos^2 C+\sin^2 C=1$.

(9) $\cos^2 D+\sin^2 D=1$.

(10) $\sin B.\cos C-\sin C.\cos B=\sin C$.

(11) $2(\cos B.\cos D+\sin B.\sin D)^2=1+\cos C$.

(12) $2(\sin D.\cos C-\sin C.\cos D)^2=1-\cos C$.

(13) $\sin 30^0=\cdot 5$. (14) $\sin 45^0=\cdot 7071....$

(15) $\sin 60^0=\cdot 8660....$ (16) $\tan 60^0=1\cdot 7320508....$

(17) $\tan 30^0=\cdot 5773....$ (18) $\sin 18^0=\cdot 3090...$

97. The actual measurement of the *line* joining two points which are any considerable distance apart, is a very tedious and difficult operation, especially when great accuracy is required; while the accurate measurement of an *angle* can, with proper instruments, be made with comparative ease and quickness.

98. A **Sextant** is an instrument for measuring the angle between the two lines drawn from the observer's eye to each of two distant objects respectively.

A **Theodolite** is an instrument for measuring angles in a horizontal plane; also for measuring '*angles of elevation*' and '*angles of depression.*'

99. The angle made with the horizontal plane, by the line joining the observer's eye with a distant object, is called

(i) its **angle of elevation,** when the object is *above* the observer;

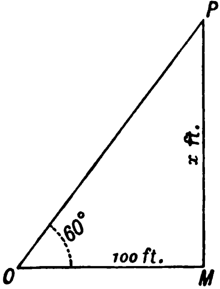
(ii) its **angle of depression,** when the object is *below* the observer. †

100. **Trigonometry** enables us by measuring certain *angles*, to deduce, from one known distance, the *lengths* of other distances: or, by the measurement of a *convenient* line, to deduce by the measurement of *angles* the lengths of lines whose actual measurement is difficult or impossible.

[In the Trigonometrical Survey of England, made by the Ordnance Department, the only distance actually measured was one of about seven miles on Salisbury Plain.]

101. For this purpose we require the numerical values of the Trigonometrical Ratios of the angles observed. Accordingly mathematical tables have been compiled, containing *lists* of the values of these Ratios. These Tables constitute a kind of numerical **Dictionary,** in which we can find the numerical value of the Trigonometrical Ratios of any required angle.

*Example* 1. At a point 100 feet from the foot of a tower the angle of elevation of the top of the tower is observed to be 60°. Find the height above the point of observation of the top of the tower.



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† In measuring the angle of *depression* the telescope is turned from a horizontal position *downwards*. See Ex. (3) p. 63.

Let $O$ be the point of observation; let $P$ be the top of the tower; let a horizontal line through $O$ meet the foot of the tower at the point $M$. Then $OM = 100$ feet, and the angle $MOP = 60^0$. Let $MP$ contain $x$ feet.

Then $$\frac{MP}{OM} = \tan MOP = \tan 60^0 = \sqrt{3}$$

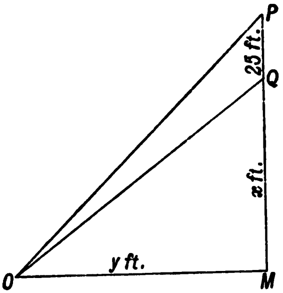
$$\therefore \frac{x}{100} = \sqrt{3}.$$

$$\therefore x = 100 . \sqrt{3} = 100 \times 1{\cdot}7320 \text{ etc.}$$

$$= 173{\cdot}2.$$

Therefore the required height is 173·2 feet.

*Example* 2. A flagstaff, 25 feet high, stands on the top of a cliff, and from a point on the seashore the angles of elevation of the highest and lowest points of the flagstaff are observed to be $47^0\ 12'$ and $45^0\ 13'$ respectively. Find the height of the cliff.



Let $O$ be the point of observation, $PQ$ the flagstaff.

Let a horizontal line through $O$ meet the vertical line $PQ$ produced in $M$.

Then $QP = 25$ feet, $MOP = 47^0\ 12'$, $MOQ = 45^0\ 13'$.

Let $MQ = x$ feet; let $OM = y$ feet.

Then $\frac{MP}{OM}=\tan 47^0\ 12'$, $\therefore\ \frac{x+25}{y}=\tan 47^0\ 12'$,

and $\frac{MQ}{OM}=\tan 45^0\ 13'$, $\therefore\ \frac{x}{y}=\tan 45^0\ 13'$.

Hence, by division $\therefore\ \frac{x+25}{x}=\frac{\tan 47^0\ 12'}{\tan 45^0\ 13'}$.

In the Tables we find that

$$\tan 47^0\ 12'=1\cdot0799018,\ \text{and}\ \tan 45^0\ 13'=1\cdot0075918,$$

$$\therefore\ 1+\frac{25}{x}=\frac{1\cdot0799018}{1\cdot0075918}=1+\frac{\cdot0723100}{1\cdot0075918},$$

$$\therefore\ \frac{x}{25}=\frac{1\cdot0075918}{\cdot0723100}=\frac{100759}{7231}.$$

$$\therefore\ x=\frac{2518975}{7231}=348\ \text{nearly}.$$

Therefore the cliff is 348 feet high.

## EXAMPLES. XIV.

The answers are given correct to three significant figures.

(1) At a point 179 feet in a horizontal line from the foot of a column, the angle of elevation of the top of the column is observed to be 45⁰. What is the height of the column?

(2) At a point 200 feet from, and on a level with the base of a tower, the angle of elevation of the top of the tower is observed to be 60⁰: what is the height of the tower?

(3) From the top of a vertical cliff, the angle of depression of a point on the shore 150 feet from the base of the cliff, is observed to be 30⁰: find the height of the cliff.

(4) From the top of a tower 117 feet high the angle of depression of the top of a house 37 feet high is observed to be 30⁰: how far is the top of the house from the tower?

✓ (5) A man 6 ft. high stands at a distance of 4 ft. 9 in. from a lamp-post, and it is observed that his shadow is 19 ft. long. Find the height of the lamp.

✓ (6) The shadow of a tower in the sunlight is observed to be 100 ft. long, and at the same time the shadow of a lamp-post 9 ft. high is observed to be $3\sqrt{3}$ ft. long. Find the angle of elevation of the sun, and the height of the tower.

✓ (7) From a point $P$ on the bank of a river, just opposite a post $Q$ on the other bank, a man walks at right angles to $PQ$ to a point $R$ so that $PR$ is 100 yards; he then observes the angle $PRQ$ to be $32^0\ 17'$: find the breadth of the river.

$$\tan 32^0\ 17' = \cdot 6317667.$$

✓ (8) A flagstaff 25 feet high stands on the top of a house; from a point on the plain on which the house stands the angles of elevation of the top and bottom of the flagstaff are observed to be $60^0$ and $45^0$ respectively: find the height of the house above the point of observation.

✓ (9) From the top of a cliff 100 feet high, the angles of depression of two ships at sea are observed to be $45^0$ and $30^0$ respectively; if the line joining the ships points directly to the foot of the cliff, find the distance between the ships.

✓ (10) A tower 100 feet high stands on the top of a cliff; from a point on the sand at the foot of the cliff the angles of elevation of the top and bottom of the tower are observed to be $75^0$ and $60^0$ respectively; find the height of the cliff. ($\text{Tan } 75^0 = 2 + \sqrt{3}$).

(11) A man walking along a straight road observes at one milestone a house in a direction making an angle $30^0$ with the road, and that at the next milestone the angle is $60^0$: how far is the house from the road?

(12) A man stands at a point $A$ on the bank $AB$ of a straight river and observes that the line joining $A$ to a post $C$ on the opposite bank makes with $AB$ an angle of $30^0$. He then goes 400 yards along the bank to $B$ and finds that $BC$ makes with $BA$ an angle of $60^0$; find the breadth of the river.

**(13) A building on a square base $ABCD$ has two of its sides, $AB$ and $CD$, parallel to the bank of a river. An observer, standing at $E$ on the other side of the river so that $DAE$ is a straight line, finds that $AB$ subtends at his eye an angle of $45^0$. Having walked $a$ yards parallel to the bank, he finds that $DE$ subtends an angle whose tangent is $\sqrt{2}$. Show that $DB=a$ yards.

(14) From the top of a hill the angles of depression of the top and bottom of a flagstaff 25 feet high at the foot of the hill are observed to be $45^0\ 13'$ and $47^0\ 12'$ respectively; find the height of the hill.

$$\tan 45^0\ 13' = 1{\cdot}0075918,$$
$$\tan 47^0\ 12' = 1{\cdot}0799018.$$

(15) From each of two stations, East and West of each other, the altitude of a balloon is observed to be $45^0$, and its bearings to be respectively N.W. and N.E.: if the stations be 1 mile apart, determine the height of the balloon.

(16) The angle of elevation of a balloon from a station due south of it is $60^0$; and from another station due west of the former and distant a mile from it it is $45^0$. Find the height of the balloon.

(17) An isosceles triangle of wood is placed on the ground in a vertical position facing the sun. If $2a$ be the base of the triangle, $b$ its height, and $30^0$ the altitude of the sun, find the tangent of half the angle at the apex of the shadow.

(18) The length of the shadow of a vertical stick is to the length of the stick as $\sqrt{3} : 1$. If the stick be turned about its lower extremity in a vertical plane, so that the shadow is always in the same direction, find what will be the angle of its inclination to the horizon when the length of the shadow is the same as before.

(19) What distance in space is travelled in an hour in consequence of the earth's rotation, by a person situated in latitude $60^0$? (Earth's radius $=4000$ miles.)

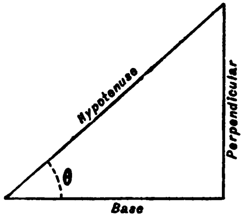
# CHAPTER VII.

## ON THE RELATIONS BETWEEN THE TRIGONOMETRICAL RATIOS OF THE SAME ANGLE.

102. THE following relations are evident from the definitions:

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}.$$

103. To prove $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.



We have $\sin\theta = \dfrac{\text{perpendicular}}{\text{hypotenuse}}$,

and $\cos\theta = \dfrac{\text{base}}{\text{hypotenuse}}$;

$$\therefore \frac{\sin\theta}{\cos\theta} = \frac{\text{perpendicular}}{\text{base}} = \tan\theta.$$

104. We may prove similarly $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

Or thus, $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$.

* 105. Euclid I. 47 gives us that in any right-angled triangle the square on the hypotenuse = the sum of the squares on the perpendicular and on the base,

or, $(\text{hypotenuse})^2 = (\text{perpendicular})^2 + (\text{base})^2$.

(i) Divide each side of this identity by $(\text{hypotenuse})^2$, and we get

$$\left(\frac{\text{hypotenuse}}{\text{hypotenuse}}\right)^2 = \left(\frac{\text{perpendicular}}{\text{hypotenuse}}\right)^2 + \left(\frac{\text{base}}{\text{hypotenuse}}\right)^2,$$

that is, $1 = \sin^2\theta + \cos^2\theta$.

(ii) Divide each side of the same identity by $(\text{base})^2$, and we get

$$\left(\frac{\text{hypotenuse}}{\text{base}}\right)^2 = \left(\frac{\text{perpendicular}}{\text{base}}\right)^2 + \left(\frac{\text{base}}{\text{base}}\right)^2,$$

that is, $\sec^2\theta = \tan^2 + 1$.

(iii) Divide each side of the same identity by $(\text{perpendicular})^2$, and we get

$$\left(\frac{\text{hypotenuse}}{\text{perpendicular}}\right)^2 = \left(\frac{\text{perpendicular}}{\text{perpendicular}}\right)^2 + \left(\frac{\text{base}}{\text{perpendicular}}\right)^2,$$

that is, $\operatorname{cosec}^2\theta = \cot^2\theta + 1$.

* 106. Thus the three results

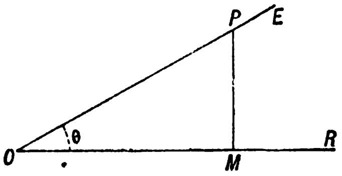
$$\left.\begin{array}{ll}\text{(i)} & \cos^2\theta + \sin^2\theta = 1 \\ \text{(ii)} & 1 + \tan^2\theta = \sec^2\theta \\ \text{(iii)} & 1 + \cot^2\theta = \operatorname{cosec}^2\theta\end{array}\right\},$$

are each a statement in Trigonometrical language of Euc. I. 47.

107. We give the above proof in a different form.

*To prove that* $\cos^2\theta + \sin^2\theta = 1$.

Let $ROE$ be any angle $\theta$.



In $OE$ take any point $P$, and draw $PM$ perpendicular to $OR$. Then with respect to $\theta$, $MP$ is the perpendicular, $OP$ is the hypotenuse, and $OM$ is the base;

$$\therefore \sin^2\theta = \frac{MP^2}{OP^2}, \quad \cos^2\theta = \frac{OM^2}{OP^2}$$

We have to prove that $\sin^2\theta + \cos^2\theta = 1$.

Now

$$\cos^2\theta + \sin^2\theta = \frac{MP^2}{OP^2} + \frac{OM^2}{OP^2}$$

$$= \frac{MP^2 + OM^2}{OP^2} = \frac{OP^2}{OP^2} = 1,$$

since by Euclid I. 47, $MP^2 + OM^2 = OP^2$,

Therefore $\cos^2\theta + \sin^2\theta = 1$.

Similarly we may prove that

$$1 + \tan^2\theta = \sec^2\theta,$$

and that $$1 + \cot^2\theta = \operatorname{cosec}^2\theta.$$

108. The following is a **List of Formulæ** with which the student must make himself familiar:

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta},$$

$$\sec\theta = \frac{1}{\cos\theta},$$

$$\cot\theta = \frac{1}{\tan\theta},$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta},$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta},$$

$$\sin^2\theta + \cos^2\theta = 1,$$

$$\tan^2\theta + 1 = \sec^2\theta,$$

$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta.$$

109. In proving Trigonometrical identities it is often convenient to express the other Trigonometrical Ratios *in terms of the sine and cosine.*

*Example. Prove that* $\tan A + \cot A = \sec A \,.\, \operatorname{cosec} A$.

Since $\tan A = \dfrac{\sin A}{\cos A}$, $\cot A = \dfrac{\cos A}{\sin A}$,

$$\sec A = \frac{1}{\cos A} \text{ and } \operatorname{cosec} A = \frac{1}{\sin A},$$

we have to prove that

$$\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} = \frac{1}{\cos A} \cdot \frac{1}{\sin A},$$

or that

$$\frac{\sin^2 A + \cos^2 A}{\cos A \,.\, \sin A} = \frac{1}{\cos A \,.\, \sin A},$$

and this is true, because $\sin^2 A + \cos^2 A = 1$.

110. Sometimes it is more convenient to express all the other Trigonometrical Ratios in terms of the *sine* only, or in terms of the *cosine* only.

*Example* (i). *Prove that*

$$\sin^4\theta + 2\sin^2\theta\cos^2\theta = 1 - \cos^4\theta.$$

Here the right-hand expression does not contain $\sin\theta$; hence for $\sin^2\theta$ in the left-hand expression we substitute $(1-\cos^2\theta)$; thus

$$\begin{aligned}\sin^4\theta + 2\sin^2\theta\cos^2\theta &= (\sin^2\theta)^2 + 2(\sin^2\theta)\cos^2\theta\\ &= (1-\cos^2\theta)^2 + 2(1-\cos^2\theta)\cos^2\theta\\ &= 1 - 2\cos^2\theta + \cos^4\theta + 2\cos^2\theta - 2\cos^4\theta\\ &= 1 - \cos^4\theta. \quad \text{Q.E.D.}\end{aligned}$$

*Example* (ii). *Prove that*

$$\sin^6\theta + \cos^6\theta = 1 - 3\cos^2\theta + 3\cos^4\theta.$$

Here

$$\begin{aligned}\sin^6\theta + \cos^6\theta &= (\sin^2\theta + \cos^2\theta)(\sin^4\theta - \sin^2\theta\cos^2\theta + \cos^4\theta)\\ &= 1 \times (\sin^4\theta - \sin^2\theta\cos^2\theta + \cos^4\theta)\\ &= (\sin^2\theta)^2 - (\sin^2\theta)\cos^2\theta + \cos^4\theta\\ &= (1-\cos^2\theta)^2 - (1-\cos^2\theta)\cos^2\theta + \cos^4\theta\\ &= 1 - 2\cos^2\theta + \cos^4\theta - \cos^2\theta + \cos^4\theta + \cos^4\theta\\ &= 1 - 3\cos^2\theta + 3\cos^4\theta. \quad \text{Q.E.D.}\end{aligned}$$

**Note.** $(1-\cos\theta)$ is called the **versed sine** of $\theta$; it is abbreviated thus versin $\theta$.

## EXAMPLES. XV.

Prove the following statements.

(1) $\cos A \,.\, \tan A = \sin A$, (2) $\cot A \,.\, \tan A = 1$.

(3) $\cos A = \sin A \,.\, \cot A$. (4) $\sec A \,.\, \cot A = \operatorname{cosec} A$.

(5) $\operatorname{cosec} A \,.\, \tan A = \sec A$.

(6) $(\tan A + \cot A) \sin A \,.\, \cos A = 1$.

(7) $(\tan A - \cot A) \sin A \,.\, \cos A = \sin^2 A - \cos^2 A$.

(8) $\cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$.

(9) $(\sin A + \cos A)^2 = 1 + 2 \sin A \,.\, \cos A$.

(10) $(\sin A - \cos A)^2 = 1 - 2 \sin A \,.\, \cos A$.

(11) $\cos^4 B - \sin^4 B = 2 \cos^2 B - 1$.

(12) $(\sin^2 B + \cos^2 B)^2 = 1$.

(13) $(\sin^2 B - \cos^2 B)^2 = 1 - 4 \cos^2 B + 4 \cos^4 B$.

(14) $1 - \tan^4 B = 2 \sec^2 B - \sec^4 B$.

(15) $(\sec B - \tan B)(\sec B + \tan B) = 1$.

(16) $(\operatorname{cosec} \theta - \cot \theta)(\operatorname{cosec} \theta + \cot \theta) = 1$.

(17) $\sin^3 \theta + \cos^3 \theta = (\sin \theta + \cos \theta)(1 - \sin \theta \cos \theta)$.

(18) $\cos^3 \theta - \sin^3 \theta = (\cos \theta - \sin \theta)(1 + \sin \theta \cos \theta)$.

(19) $\sin^6 \theta + \cos^6 \theta = 1 - 3 \sin^2 \theta \,.\, \cos^2 \theta$.

(20) $(\sin^6 \theta - \cos^6 \theta) = (2 \sin^2 \theta - 1)(1 - \sin^2 \theta + \sin^4 \theta)$.

(21) $\dfrac{\tan A + \tan B}{\cot A + \cot B} = \tan A \,.\, \tan B$.

(22) $\dfrac{\cot \alpha + \tan \beta}{\tan \alpha + \cot \beta} = \cot \alpha \,.\, \tan \beta$.

(23) $\dfrac{1 - \sin A}{1 + \sin A} = (\sec A - \tan A)^2$.

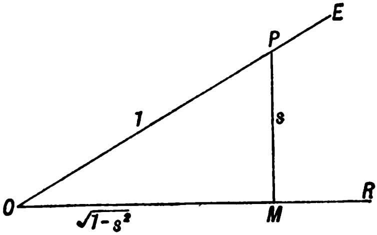
(24) $\dfrac{1 + \cos A}{1 - \cos A} = (\operatorname{cosec} A + \cot A)^2$.

(25) $2 \operatorname{versin} \theta - \operatorname{versin}^2 \theta = \sin^2 \theta$.

(26) $\operatorname{versin} \theta \,(1 + \cos \theta) = \sin^2 \theta$.

**111.** All the Trigonometrical Ratios of an angle can be expressed in terms of any one of them.

*Example* 1. *To express all the trigonometrical ratios of an angle in terms of the sine.*



Let $ROE$ be any angle $A$.

We can take $P$ anywhere in the line $OE$; so that we can make *one* of the lines, $OP$, $OM$, or $MP$ any length we please.

Let us take $OP$ so that its measure is 1, and let $s$ be the measure of $MP$; so that $\sin A$, which is $\frac{MP}{OP}$, $=\frac{s}{1}$; or, $s=\sin A$.

Let $x$ be the measure of $OM$.

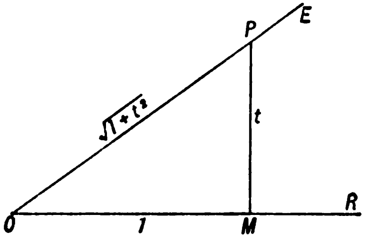
Then since
$$OM^2=OP^2-MP^2,$$
$$\therefore\ x^2=1-s^2,$$
$$\therefore\ x=\sqrt{1-s^2},$$

Hence
$$\cos A=\frac{OM}{OP}=\frac{\sqrt{1-s^2}}{1}=\sqrt{1-\sin^2 A},$$
$$\tan A=\frac{MP}{OM}=\frac{s}{\sqrt{1-s^2}}=\frac{\sin A}{\sqrt{1-\sin^2 A}},$$

and so on.

Note. The solution of the equation $x^2=1-s^2$, gives
$$x=\pm\sqrt{1-s^2},$$
and therefore the ambiguity ($\pm$) must stand before each of the root symbols in the above. This ambiguity, as will be explained later on, is of great use when the magnitude of the angle is not limited. When we limit $A$ to be less than a right angle we have no use for the negative sign.

*Example* 2. To express all the other trigonometrical ratios of an angle in terms of the tangent.



In this case $\tan\theta = \frac{MP}{OM}$.

Take $P$ so that the measure of $OM$ is 1, and let $t$ be the measure of $MP$; so that $\tan\theta$, which is $\frac{MP}{OM}$, $= \frac{t}{1}$; or, $t = \tan\theta$.

Then we can show that the measure of $OP$ is $\sqrt{1+t^2}$.

Hence, $$\sin\theta = \frac{MP}{OP} = \frac{t}{\sqrt{1+t^2}} = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}.$$

$$\cos\theta = \frac{OM}{OP} = \frac{1}{\sqrt{1+t^2}} = \frac{1}{\sqrt{1+\tan^2\theta}}.$$

and so on.

112. The same results may be obtained by the use of the formulæ on page 69.

*Example* $$\cos^2\theta + \sin^2\theta = 1,$$

$$\therefore \cos^2\theta = 1 - \sin^2\theta,$$

$$\therefore \cos\theta = \sqrt{1-\sin^2\theta}.$$

Again $$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}},$$

and so on.

## EXAMPLES. XVI.

(1) Express all the other Trigonometrical Ratios of $A$ in terms of $\cos A$.

(2) Express all the other Trigonometrical Ratios of $A$ in terms of $\cot A$.

(3) Express all the other Trigonometrical Ratios of $A$ in terms of $\sec A$.

(4) Express all the other Trigonometrical Ratios of $A$ in terms of $\operatorname{cosec} A$.

(5) Use the formulæ of page 69 to express all the other Trigonometrical Ratios of $A$ in terms of $\sin A$.

(6) Use the formulæ of page 69 to express all the other Trigonometrical Ratios of $A$ in terms of the $\tan A$.

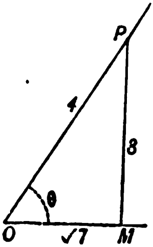
113. *Given one of the Trigonometrical Ratios of an angle less than a right angle, we can find all the others.*

Since all the Trigonometrical Ratios of an angle can be expressed in terms of any one of them, it is clear that if the numerical value of any one of them be given, the numerical value of all the rest can be found.

*Example.* Given $\sin \theta = \frac{3}{4}$, find the other Trigonometrical Ratios of $\theta$.

Let $ROE$ be the angle $\theta$.

Take $P$ on $OE$ so that the measure of $OP$ is 4. Draw $PM$ perpendicular to $OR$.



Then since $\sin\theta = \frac{3}{4}$ $\left(\text{so that } \frac{MP}{OP} = \frac{3}{4}\right)$, and since the measure of $OP$ is 4, therefore the measure of $MP$ must be 3.

Let $x$ be the measure of $OM$;

then
$$OM^2 = OP^2 - MP^2,$$
$$\therefore\ x^2 = 4^2 - 3^2 = 16 - 9 = 7.$$
$$x = \sqrt{7}.$$

Therefore the measure of $OM$ is $\sqrt{7}$.

Hence,
$$\cos\theta = \frac{OM}{OP} = \frac{\sqrt{7}}{4},$$
$$\tan\theta = \frac{MP}{OM} = \frac{3}{\sqrt{7}} = \frac{3\sqrt{7}}{7}.$$
$$\cot\theta = \frac{\sqrt{7}}{3}.$$

## EXAMPLES. XVII.

(1) Given that $\sin A = \frac{3}{5}$, find $\tan A$ and $\operatorname{cosec} A$.

(2) Given that $\cos B = \frac{1}{3}$, find $\sin B$ and $\cot B$.

(3) Given that $\tan A = \frac{4}{3}$, find $\sin A$ and $\sec A$.

(4) Given that $\sec\theta = 4$, find $\cot\theta$ and $\sin\theta$.

(5) Given that $\tan\theta = \sqrt{3}$, find $\sin\theta$ and $\cos\theta$.

(6) Given that $\cot\theta = \frac{2}{\sqrt{5}}$, find $\sin\theta$ and $\sec\theta$.

(7) Given that $\sin\theta = \frac{b}{c}$, find $\tan\theta$.

(8) Given that $\tan\theta = \frac{a}{b}$, find $\sin\theta$ and $\cos\theta$.

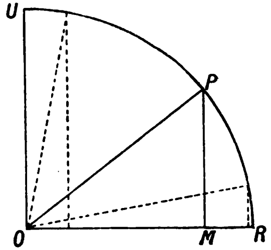
(9) Given that $\cos\theta = \frac{1}{a}$, find $\sin\theta$ and $\cot\theta$.

(10) If $\sin\theta = a$, and $\tan\theta = b$, prove that $(1 - a^2)(1 + b^2) = 1$.

(11) If $\cos\theta = h$, and $\tan\theta = k$, find the equation connecting $h$ and $k$.

114. The following propositions are important.

PROP. I. *To trace the changes in the magnitude of* sin A *as* A *increases from* $0^0$ *to* $90^0$.



Take a line $OR$, of any length; and describe the quadrant $RPU$ of the circle whose centre is $O$ and radius $OR$.

Draw the right angle $ROU$, cutting the circle in $U$.

Let $OP$ make any angle $ROP$ $(=A)$ with $OR$; draw $PM$ perpendicular to $OR$.

Then $$\sin A = \frac{MP}{OP}.$$

When the angle $A$ is $0^0$, $MP$ is zero, and when $A$ is $90^0$, $MP$ is equal to $OP$; and as $A$ continuously increases from $0^0$ to $90^0$, $MP$ increases continuously from zero to $OP$; also $OP$ is always equal to $OR$.

Therefore, when $A=0^0$, the fraction $\frac{MP}{OP}$ is equal to $\frac{0}{OP}$, that is 0; when $A=90^0$ the fraction $\frac{MP}{OP}$ is equal to $\frac{OP}{OP}$, that is 1; and as $A$ continuously increases from $0^0$ to $90^0$, the *numerator* of the fraction $\frac{MP}{OP}$ *continuously increases* from zero to $OP$, while the denominator is unchanged, and therefore the *fraction* $\frac{MP}{OP}$, which is sin $A$, *increases continuously* from 0 to 1.

PROP. II. *To trace the changes in the magnitude of* tan A *as* A *increases from* 0° *to* 90°.

With the same construction and figure as in the last article, we have

$$\tan A = \frac{MP}{OM}.$$

When the angle $A$ is $0^0$, $MP$ is zero; when $A$ is $90^0$, $MP$ is equal to $OP$; and as the angle continuously increases from $0^0$ to $90^0$, $MP$ increases continuously from zero to $OP$.

When the angle $A$ is $0^0$, $OM$ is equal to $OP$; when $A$ is $90^0$, $OM$ is zero; and as $A$ continuously increases from $0^0$ to $90^0$, $OM$ continuously decreases from $OP$ to zero.

Hence, when $A$ is $0^0$, the fraction $\frac{MP}{OM}$ is equal to $\frac{0}{OM}$, that is 0; when $A$ is $90^0$, the fraction $\frac{MP}{OM}$ is equal to $\frac{OP}{0}$, that is 'infinity' (see art. 95); and as $A$ continuously increases from $0^0$ to $90^0$, the numerator continuously increases from zero to $OP$, while the denominator continuously diminishes from $OP$ to zero; so that the fraction $\frac{MP}{OM}$, which is $\tan A$, *continuously* increases from 0 until it is *greater than any assignable numerical quantity.*

## EXAMPLES. XVIII.

(1) Show that as $A$ continuously increases from $0^0$ to $90^0$, $\cos A$ continuously diminishes from 1 to 0.

(2) Trace the changes in the magnitude of $\sec\theta$ as $\theta$ increases from 0 to $\frac{\pi}{2}$.

(3) Trace the changes in the magnitude of $\sin A$ as $A$ diminishes from $90^0$ to $0^0$.

(4) Trace the changes in the magnitude of $\cot\theta$ as $\theta$ increases from 0 to $\frac{\pi}{2}$.

115. Since the hypotenuse is the *greatest* side in a right-angled triangle, it is clear

(i) that *sin A*, which is $\frac{\text{perpendicular}}{\text{hypotenuse}}$, is never greater than unity,

(ii) that *cos A*, which is $\frac{\text{base}}{\text{hypotenuse}}$, is never greater than unity,

(iii) that *cosec A*, which is $\frac{\text{hypotenuse}}{\text{perpendicular}}$, is never *numerically* less than unity (we shall see later on that it may be *negative*),

(iv) that *sec A*, which is $\frac{\text{hypotenuse}}{\text{base}}$, is never numerically less than unity (it may be negative).

Hence such a question as '*Find an angle whose* **sine** *is* $\frac{4}{3}$' has no solution, since there is no angle whose sine is greater than unity.

And such a question as '*Find an angle whose* **secant** *is* $\frac{3}{4}$' has no solution, because there is no angle whose secant is numerically less than unity.

116. If *A* be small, the perpendicular is smaller than the base; and tan *A*, which is $\frac{\text{perpendicular}}{\text{base}}$, can be made as small as we please (see figure in Art. 114). Also if the angle *A* be nearly a right angle, the perpendicular is greater than the base; and tan *A*, which is $\frac{\text{perpendicular}}{\text{base}}$, can be made as large as we please.

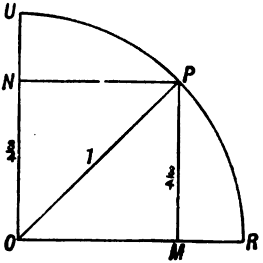
So that an angle whose tangent is $a$ can *always* be found if $a$ be positive; that is, there is always an angle $A$ between 0° and 90°, such that $\tan A = a$, if $a$ be a positive number.

[The student will find as he proceeds that the equation $\tan x = a$ can always be solved if $a$ be any *arithmetical* quantity].

*Example* 1. *To draw an angle* A *whose sine is* $\frac{3}{4}$.

Draw any line $OR$, and take $R$ such that the measure of $OR$ is unity; describe the quadrant $RPU$ of the circle whose centre is $O$ and radius $OR$.

In $OU$ which is perpendicular to $OR$, take $ON$ so that the measure of $ON$ is $\frac{3}{4}$.



Draw $NP$ parallel to $OR$ cutting the quadrant in $P$. (The student should observe that if the measure of $ON$ were *greater than unity*, $N$ would be outside the circle altogether, and the line parallel to $OR$ would not cut the circle at all.)

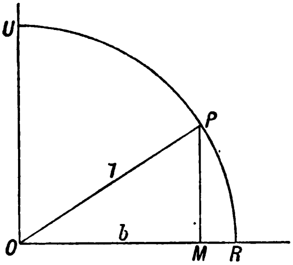
Join $OP$, and draw $PM$ perpendicular to $OR$.

Then $ROP$ is the angle required.

For $\sin ROP = \dfrac{MP}{OP} = \dfrac{ON}{OP} = \frac{3}{4} \div 1 = \frac{3}{4}$.

Therefore an angle $POR$ has been drawn whose sine is $\frac{3}{4}$.

*Example* 2. *To draw an angle whose cosine is* b; where $b$ is a *proper* fraction and positive.



Draw any line $OR$, and take $OR$ so that its measure is unity.

Describe the quadrant $RPU$ of the circle whose centre is $O$ and radius $OR$.

Take $M$ in $OR$ so that the measure of $OM$ is $b$. $OM$ will be less than $OR$ because $b$ is a *proper* fraction.

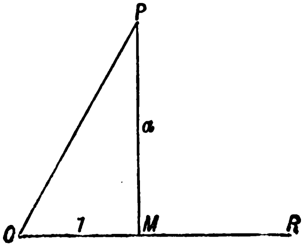
Draw $MP$ perpendicular to $OR$ cutting the quadrant in $P$. (The student will observe that if $b$ were an *improper* fraction, $M$ would be outside the circle altogether.)

Then the angle $ROP$ is the angle required.

For $\cos ROP = \dfrac{OM}{OP} = \dfrac{b}{1} = b$.

Therefore an angle $ROP$ has been described such that the $\cos ROP = b$.

*Example* 3. *To draw an angle A such that* $\tan A = a$, where $a$ is any positive numerical quantity.



Draw any line $OR$, and take $M$ in it so that the measure of $OM$ is unity.

Draw $MP$ perpendicular to $OM$, and take $P$ so that the measure of $MP$ is $a$. Join $OP$. Then the angle $ROP$ is the angle required.

For $\tan ROP = \dfrac{MP}{OM} = \dfrac{a}{1} = a$.

Thus an angle $ROP$ has been drawn so that $\tan ROP = a$.

117. Definition. One angle is called the **complement** of another, when the two angles added together make up a *right angle*.

*Example* 1. The complement of $A$ is $(90^\circ - A)$.

*Example* 2. The complement of $190^\circ$ is $(90^\circ - 190^\circ) = -100^\circ$.
For $190^\circ + (90^\circ - 190^\circ) = 90^\circ$.

*Example* 3. The complement of $\frac{5\pi}{4}$ is $\left(\frac{\pi}{2} - \frac{5\pi}{4}\right) = -\frac{3\pi}{4}$.

118. *To prove that the sine of an angle* A *is equal to the cosine of its complement* $(90^\circ - A)$.

[This is true whatever be the magnitude of $A$, as will be proved later on.]

If $A$ be less than $90^\circ$, let $ROP$ be $A$ (see last figure).

Draw $PM$ perpendicular to $OR$.

Then since $PMO = 90^\circ$, therefore $POM + OPM = 90^\circ$ and therefore $OPM = (90^\circ - A)$.

Now, $\sin A = \dfrac{MP}{OP} = \cos OPM = \cos (90^\circ - A)$. Q.E.D.

Ratio of angle is the co-ratio of its complements

## EXAMPLES. XIX.

(1) Draw an angle whose sine is $\frac{1}{2}$.

(2) Draw an angle whose cosecant is 2.

(3) Draw an angle whose tangent is 2.

(4) Can an angle be drawn whose tangent is 431 ?

(5) Can an angle be drawn whose cosine is $\frac{4}{3}$ ?

(6) Can an angle be drawn whose secant is 5 ?

(7) Find the complements of

(i) $30^0$. (ii) $190^0$. (iii) $90^0$. (iv) $350^0$.

(v) $-25^0$. (vi) $-320^0$. (vii) $\frac{3\pi}{4}$. (viii) $-\frac{\pi}{6}$.

(8) Prove that $\sin 70^0 = \cos 20^0$.

(9) Prove that $\cos 47^0\, 16' = \sin 42^0\, 44'$.

(10) Prove that $\tan 79^0 = \cot 11^0$.

(11) Prove that $\sec 36^0 = \operatorname{cosec} 54^0$.

(12) If $A$ be less than $90^0$, prove

(i) $\cos A = \sin (90^0 - A)$.
(ii) $\tan A = \cot (90^0 - A)$.
(iii) $\sec A = \operatorname{cosec} (90^0 - A)$.
(iv) $\cot A = \tan (90^0 - A)$.

## ON THE SOLUTION OF TRIGONOMETRICAL EQUATIONS.

119. A TRIGONOMETRICAL equation is an equation in which there is a letter, such as $\theta$, which stands for an *angle* of unknown magnitude.

The *solution* of the equation is the process of finding an angle which, if it be substituted for $\theta$, satisfies the equation.

*Example* 1. *Solve* $\cos\theta = \frac{1}{2}$.

This is a Trigonometrical equation. To solve it we must find some angle such that its cosine is $\frac{1}{2}$.

We know that $\cos 60^0 = \frac{1}{2}$.

Therefore if $60^0$ be put for $\theta$ the equation is satisfied.

$\therefore$ $\theta = 60^0$ is a solution of the equation.

*Example* 2. *Solve* $\sin\theta - \operatorname{cosec}\theta + \frac{3}{2} = 0$.

The usual method of solution is to express all the Trigonometrical Ratios in terms of one of them.

Thus we put $\frac{1}{\sin\theta}$ for $\operatorname{cosec}\theta$, and we get

$$\sin\theta - \frac{1}{\sin\theta} + \frac{3}{2} = 0.$$

This is an equation in which $\theta$, and therefore *sin* $\theta$ is unknown. It will be convenient if we put $x$ for *sin* $\theta$, and then solve the equation for $x$ as an ordinary algebraical equation. Thus we get

$$x - \frac{1}{x} + \frac{3}{2} = 0,$$

or,
$$x^2 + \frac{3x}{2} = 1,$$

$$\therefore\ x^2 + \frac{3x}{2} + \frac{9}{16} = 1 + \frac{9}{16} = \frac{25}{16}.$$

$$\therefore\ x + \frac{3}{4} = \pm\frac{5}{4}.$$

Whence
$$x = -2, \text{ or } x = \frac{1}{2}.$$

But $x$ stands for $\sin\theta$.

Thus we get
$$\sin\theta = -2, \text{ or } \sin\theta = \frac{1}{2}.$$

The value $-2$ is inadmissible for $\sin\theta$, for there is no angle whose sine is numerically greater than 1 (Art. 115).

$$\therefore\ \sin\theta = \frac{1}{2}.$$

But
$$\sin 30^0 = \frac{1}{2}.$$

$$\therefore\ \sin\theta = \sin 30^0.$$

Therefore *one* angle which satisfies this equation for $\theta$ is $30^0$.

*Example* 3. To solve the equation $\operatorname{cosec}\theta - \cot^2\theta + 1 = 0$.

We have $1 + \cot^2\theta = \operatorname{cosec}^2\theta$. [Art. 108]

$$\therefore \cot^2\theta = \operatorname{cosec}^2\theta - 1.$$

and we get $\operatorname{cosec}\theta - \operatorname{cosec}^2\theta + 2 = 0$.

Let $x$ stand for *cosec* $\theta$. Then

$$x - x^2 + 2 = 0.$$

$$\therefore x^2 - x = +2.$$

or $\therefore x^2 - x + \frac{1}{4} = +\frac{1}{4} + 2 = \frac{9}{4}$.

$$\therefore x - \frac{1}{2} = \pm\frac{3}{2}.$$

Whence $x = 2$, or $x = -1$,

$$\therefore \operatorname{cosec}\theta = 2,$$

but $\operatorname{cosec} 30^0 = 2$,

$$\therefore \operatorname{cosec}\theta = \operatorname{cosec} 30^0.$$

therefore $30^0$ is *one* angle which satisfies the equation for $\theta$.

## EXAMPLES. XX.

Find one angle which satisfies each of the following equations.

[illegible]
5
to
12

(1) $\sin\theta = \frac{1}{\sqrt{2}}$.

(2) $4\sin\theta = \operatorname{cosec}\theta$.

(3) $2\cos\theta = \sec\theta$.

(4) $4\sin\theta - 3\operatorname{cosec}\theta = 0$.

(5) $4\cos\theta - 3\sec\theta = 0$.

(6) $3\tan\theta = \cot\theta$.

(7) $3\sin\theta - 2\cos^2\theta = 0$.

(8) $\sqrt{2}\sin\theta = \tan\theta$.

(9) $2\cos\theta = \sqrt{3}\cot\theta$.

(10) $\tan\theta = 3\cot\theta$.

(11) $\tan\theta + 3\cot\theta = 4$.

(12) $\tan\theta + \cot\theta = 2$.

(13) $2\sin^2\theta + \sqrt{2}\cos\theta = 2$.

(14) $4\sin^2\theta + 2\sin\theta = 1$.

(15) $3\tan^2\theta - 4\sin^2\theta = 1$.

(16) $2\sin^2\theta + \sqrt{2}\sin\theta = 2$.

(17) $\cos^2\theta - \sqrt{3}\cos\theta + \frac{3}{4} = 0$.

(18) $\cos^2\theta + 2\sin^2\theta - \frac{5}{2}\sin\theta = 0$.

## * * MISCELLANEOUS EXAMPLES. XXI.

(1) Prove that $3\sin 60^\circ - 4\sin^3 60^\circ = 4\cos^3 30^\circ - 3\cos 30^\circ$.

(2) Prove that $\tan 30^\circ (1 + \cos 30^\circ + \cos 60^\circ) = \sin 30^\circ + \sin 60^\circ$.

(3) If $2\cos^2\theta - 7\cos\theta + 3 = 0$, show there is only one value of $\cos\theta$.

(4) Find $\cos\theta$ from the equation $8\cos^2\theta - 8\cos\theta + 1 = 0$.

(5) Find $\sin\theta$ from the equation $8\sin^2\theta - 10\sin\theta + 3 = 0$, and prove that one value of $\theta$ is $\frac{\pi}{6}$.

(6) Find $\tan\theta$ from the equation $12\tan^2\theta - 13\tan\theta + 3 = 0$.

(7) If $3\cos^2\theta + 2 . \sqrt{3} . \cos\theta = 5\frac{1}{4}$, show that there is only one value of $\cos\theta$, and that one value of $\theta$ is $\frac{\pi}{6}$.

(8) Prove that the value of $\sin^4\theta + \cos^4\theta + 2 . \sin^2\theta . \cos^2\theta$ is always the same.

(9) Simplify $\cos^4 A + 2 . \sin^2 A . \cos^2 A$.

(10) Express $\sin^6 A + \cos^6 A$ in terms of $\sin^2 A$ and powers of $\sin^2 A$.

(11) Express $1 + \tan^4\theta$ in terms of $\cos\theta$ and its powers.

(12) Prove that $\frac{\cos A + \cos B}{\sin A - \sin B} + \frac{\sin A + \sin B}{\cos A - \cos B} = 0$.

(13) Express $(\sec A - \tan A)^2$ in terms of $\sin A$.

(14) Trace the changes in the magnitude of $\operatorname{cosec}\theta$ as $\theta$ increases from 0 to $\frac{\pi}{2}$.

(15) Trace the changes in the magnitude of $\cot\theta$ as $\theta$ decreases from $\frac{\pi}{2}$ to 0.

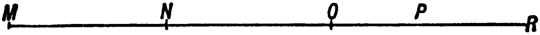
(16) Solve $\sin(\theta + \phi) = \frac{\sqrt{3}}{2}$, $\cos(\theta - \phi) = \frac{\sqrt{3}}{2}$.

## CHAPTER VIII.

### ON THE USE OF THE SIGNS + AND –.

120. THE student is probably aware that, in the application of Algebra to Problems concerning *distance*, we sometimes find that the solution of an equation gives the measure of a *distance* with the sign – before it.

*Example.* Let $M$, $N$, $O$ be places in a straight line; let the distance from $M$ to $N$ be 3 miles, and the distance from $N$ to $O$, 3 miles.



One man $A$ starting from $M$, rides towards $O$ at the rate of 10 miles an hour, while another man $B$ starting simultaneously from $N$, walks towards $O$ at the rate of 4 miles an hour;

If $Q$ be the point at which they meet, *how far is $Q$ beyond $O$?*

Let $P$ be any point beyond $O$, and let $x$ be the number of miles in $OP$. We wish to find $x$, i.e. the measure of $OP$, so that $P$ may coincide with $Q$, the point at which $A$ overtakes $B$.

When $A$ arrives at $P$, he has ridden $6+x$ miles. The time occupied at the rate of 10 miles an hour is $\frac{6+x}{10}$ hours.

When $B$ arrives at $P$, he has walked $3+x$ miles. The time occupied at the rate of 4 miles an hour is $\frac{3+x}{4}$ hours.

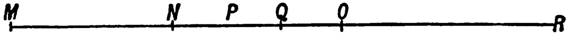
When $P$ is the point at which they meet, these times are equal, so that

$$\frac{6+x}{10}=\frac{3+x}{4}; \quad \text{whence } x=-1.$$

Thus the required *number of miles* has the sign - before it; and we have failed to find a point *beyond* $O$ at which $A$ overtakes $B$.

121. Such a result can generally be *interpreted* by altering the statement of the problem, thus:

The line whose measure appears with the sign - before it, must be considered as drawn in the direction opposite to that in which it was drawn in the first statement of the problem.



*Example.* Taking the former example, let us alter the question as follows:

If $Q$ be the point at which $A$ overtakes $B$, how far is $Q$ to the left of $O$?

Let $P$ be any point *to the left* of $O$, and let $x$ be the number of miles in $OP$.

We wish to find $x$ (i.e. the measure of $OP$), so that $P$ may coincide with $Q$, the point at which $A$ overtakes $B$.

When $A$ arrives at $P$, he has ridden $6-x$ miles.

When $B$ arrives at $P$, he has walked $3-x$ miles.

Proceeding as before, we get

$$\frac{6-x}{10}=\frac{3-x}{4}; \quad \text{whence } x=+1.$$

Therefore if $P$ is to coincide with $Q$ (the point at which $A$ overtakes $B$), $OP$ must be one mile *to the left* of $O$.

122. The consideration of such examples as the above has suggested, that the sign – may be made use of, in the application of Algebra to Geometry, to represent a **direction** exactly opposite to that represented by the sign +.

Accordingly the following Rule, or Convention, has been made.

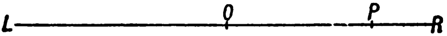
~~RULE.~~ Any straight line $AB$ being given, then

lines drawn parallel to $AB$ in **one direction** shall be **positive**; that is, shall be *represented algebraically* by their *measures with the sign + before them:*

lines drawn parallel to $BA$ in the **opposite direction** shall be **negative**; that is, shall be *represented algebraically* by their *measures with the sign – before them.*

123. We may choose for the positive direction in each case that direction which is most convenient.

*Example.* Let $LR$ be a straight line parallel to the printed lines in the page,



and let lines drawn in the direction from $L$ to $R$ in the figure that is, from the left-hand towards the right, be considered *positive.* Then by the above rule, lines drawn in the direction from $R$ to $L$, that is, from right to left, must be *negative.*

124. In naming a line by the letters at its extremities, we can indicate by the **order of the letters**, the direction in which the line is supposed to be drawn.

*Example.* Let $O$ and $P$ be two points in the line $LR$ as in the figure, and let the measure of the distance between them be $a$.

Then $OP$, i.e. the line drawn from $O$ to $P$, which is in the *positive* direction, is *represented algebraically* by $+a$.

While $PO$, i.e. the line drawn from $P$ to $O$, which is in the *negative* direction, is *represented algebraically* by $-a$.

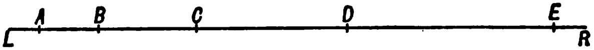
125. Hence in using the two letters at its extremities to represent a line, the student will find it advantageous always to pay careful attention to the **order of the letters.**

*Example.* Let $LR$ be a straight line parallel to the printed lines in the page.

Let $A$, $B$, $C$, $D$, $E$ be points in $LR$, such that the measures of $AB$, $BC$, $CD$, $DE$, are 1, 2, 3, 4 respectively.

Find the algebraical representation of

(i) $AC+CB$ (ii) $AD+DC-BC$.



(i) The algebraical representation of $AC$ is $+3$,

the algebraical representation of $CB$ is $-2$.

Hence that of $AC+CB$ is $+3-2$; that is, $+1$*.

(ii) The algebraical representation of $AD$ is $+6$, that of $DC$ is $-3$, and that of $BC$ is $+2$.

Therefore that of $AD+DC-BC=6-3-2=+1$.

This is equivalent to that of $AB$.

## EXAMPLES. XXII.

In the above figure, find the algebraical representation of

(1) $AB+BC+CD$. (2) $AB+BC+CA$.

(3) $BC+CD+DE+EC$. (4) $AD-CD$.

(5) $AD+DB+BE$. (6) $BC-AC+AD-BD$.

(7) $CD+DB+BE$. (8) $CD-BD+BA+AC+CE$.

---

* By $AC+CB$ (attention being paid to *direction*), we mean 'Go from $A$ to $C$ and from $C$ to $B$.' The result is equivalent to starting from $A$ and stopping at $B$, *i.e.* equivalent to $AB$.

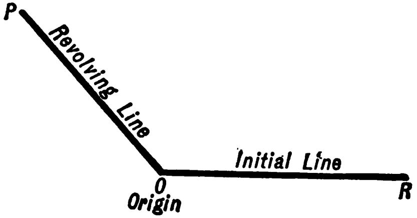
## CHAPTER IX.

### ON THE USE OF THE SIGNS + AND – IN TRIGONOMETRY.

126. In **Trigonometry** in order conveniently to treat of angles of *any* magnitude, we proceed as follows.

We take a fixed point $O$, called the **origin**; and a fixed straight line $OR$, called the **initial line.**

The angle of which we wish to treat is described by a line $OP$, called the **revolving line.** This line $OP$ *starts from* the initial line $OR$, and turns about $O$ through an angle $ROP$ of any proposed magnitude into the position $OP$.



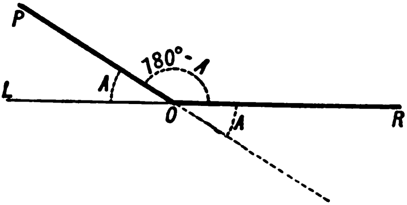
127. We have already said in Art. 41

(i) that, when an angle $ROP$ is described by $OP$ turning about $O$ in the direction *contrary to that of the hands of a watch*, the angle $ROP$ is said to be **positive**; that is, *is represented algebraically* by its *measure with the sign + before it.*

(ii) that, when an angle $ROP$ is described by $OP$ turning about $O$ in the *same* direction as the hands of a watch, the angle is said to be **negative**; that is, is *represented algebraically* by its *measure with the sign – before it.*

*Example.* $(180^0 - A)$ indicates

(i) the angle described by $OP$ turning about $O$ from the position $OR$ in the positive direction until it has described an angle of $(180 - A)$ degrees.



Or, (ii) the angle described by $OP$ turning about $O$, from the position $OR$, in the positive direction until it has described an angle of $180^0$ (when it has turned into the position $OL$), and then turning back from $OL$ in the negative direction through the angle $-A$ into the position $OP$.

Or, (iii) the angle described by $OP$ turning about $O$ from the position $OR$, in the negative direction through the angle $-A$, and then turning back in the positive direction through the angle $180^0$, into the position $OP$.

The student should observe that in each of these three ways of regarding the angle $(180^0 - A)$, the *resulting* angle $ROP$ is the same.

## EXAMPLES. XXIII.

Draw a figure giving the position of the revolving line after it has turned through each of the following angles.

(1) $270^0$. (2) $370^0$. (3) $425^0$. (4) $590^0$.

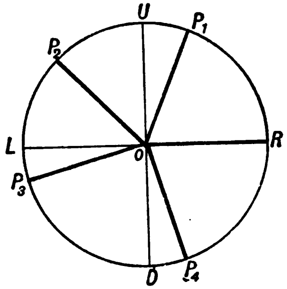
(5) $-30^0$. (6) $-330^0$. (7) $-480^0$. (8) $-750^0$.

(9) $\frac{27\pi}{4}$. (10) $2n\pi + \frac{\pi}{6}$. (11) $(2n+1)\pi + \frac{\pi}{3}$.

(12) $(2n+1)\pi - \frac{\pi}{4}$. (13) $2n\pi - \frac{\pi}{2}$. (14) $(2n+1)\pi - \frac{\pi}{2}$.

Note. $n\pi$ always stands for a whole number of two right angles.

128. It is often convenient to keep the revolving line of the same length.



In this case the point $P$ lies always on the circumference of a circle whose centre is $O$.

Let this circle cut the lines $LOR$, $UOD$ in the points $L$, $R$, $U$, $D$ respectively.

The circle $RULD$ is thus divided at the points $R$, $U$, $L$, $D$ into four **Quadrants**, of which

$RU$ is called the **first Quadrant.**

$UL$ is called the **second Quadrant.**

$LD$ is called the **third Quadrant.**

$DR$ is called the **fourth Quadrant.**

Hence, in the figure,

$ROP_1$ is an angle of the first Quadrant.

$ROP_2$ ,, ,, second Quadrant.

$ROP_3$ ,, ,, third Quadrant.

$ROP_4$ ,, ,, fourth Quadrant.

129. When we are given the initial and the final line of an angle, we can at once decide the Quadrant of which it is. We cannot however decide the **magnitude** of the *trigonometrical* angle.

For we do not know how many **complete** revolutions the revolving line may have made.

In other words, when the *geometrical* representation of an angle (consisting of the positions of the initial line and of the final line) is given, we are *not* (without further information) given the *Trigonometrical* angle.

The Geometrical angle is here taken to be the amount of turning in the positive direction, which will transfer the revolving line from the initial to the final line.

130. Let $A^\circ$ be the magnitude of a given geometrical angle $ROP$; then, every **trigonometrical** angle having $OR$ for its initial line and $OP$ for its final line, is included in the expression

$$n \times \mathbf{360^\circ} + \mathbf{A^\circ}; \quad \text{or,} \quad \mathbf{2}n \times \mathbf{180^\circ} + \mathbf{A^\circ},$$

where $n$ is some integer positive or negative.

For $n \times 360^\circ$ indicates a number of *complete* revolutions of the revolving line.

When we know the *magnitude* of the trigonometrical angle, then we know what integer $n$ is.

*Example.* In the figure on p. 92 the geometrical angle $ROP_2$ contains 135°. This angle is the geometrical representation of the angles

$$495^\circ, \; 855^\circ, \; -225^\circ, \text{ etc.},$$

for $495^\circ = 360^\circ + 135^\circ$; $855^\circ = 720^\circ + 135^\circ$; $-225^\circ = -360^\circ + 135^\circ$.

N.B. Let $\theta$ be the circular measure of a geometrical angle $ROP$; then, every trigonometrical angle represented by $ROP$, is included in the expression

$$n \times \mathbf{2}\pi + \theta; \quad \text{or,} \quad \mathbf{2}n\pi + \theta.$$

## EXAMPLES. XXIV.

**State in which Quadrant the revolving line will be after describing the following angles:**

(1) 120°. (2) 340°. (3) 490°. (4) −100°.

(5) −380°. (6) −1000°. (7) $\frac{2\pi}{3}$. (8) $10\pi+\frac{\pi}{4}$.

(9) $9\pi-\frac{3\pi}{4}$. (10) $2n\pi-\frac{\pi}{4}$. (11) $(2n+1)\pi+\frac{2\pi}{3}$. (12) $n\pi+\frac{\pi}{6}$.

Shew that the following angles are represented by the same diagram geometrically :

(13) 365°, 725°, −355°, 1085°, −715°.
(14) −65°, 295°, 655°, −425°.
(15) 199°, 559°, 3799°, −521°, 7001°.
(16) $\theta$, $4\pi+\theta$, $-(2\pi-\theta)$, $-4\pi+\theta$.
(17) $\pi-a$, $3\pi-a$, $-\pi-a$, $(2n+1)\pi-a$.
(18) $\pi+a$, $3\pi+a$, $-\pi+a$, $(2n+1)\pi+a$.

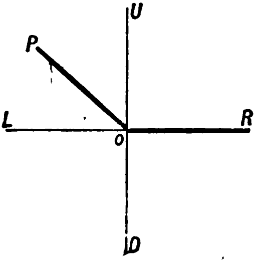
131. The principal **directions** of lines with which we are concerned in **Trigonometry** are as follows ;

I. that parallel to *OR*, the initial line.

*OR* is usually drawn from *O* towards the right hand, parallel to the printed lines of the page.

II. The direction at right angles to *OR*.

III. The direction parallel to the revolving line *OP*.



Accordingly we make the following **rules;**

I. For lines parallel to the initial line $OR$, the **positive** direction is from $O$ to $R$; consequently the **negative** direction is from $R$ to $O$.

II. $OU$ **is the position of the revolving line after it has described from** $OR$ **a right angle about** $O$ **in the positive direction of revolution;** and for lines perpendicular to $OR$, the **positive** direction is from $O$ to $U$; consequently for lines perpendicular to $OR$ the **negative** direction is from to $U$ to $O$.

III. Any line drawn parallel to the revolving line in the direction from $O$ to $P$ is to be **positive,** and consequently any line drawn in the direction from $P$ to $O$ is to be negative.

Note. The student must notice that the revolving line $OP$ *carries its positive direction round with it*, so that the line '$OP$' is *always positive*. The revolving line, as it turns about $O$, nowhere undergoes any *sudden* change of direction such as would be indicated by a change of sign.

132. We said, in Art. 81, that **the definitions of the Trigonometrical Ratios** (on pp. 46, 47), **apply to angles of any magnitude.** We have only to remark that it is generally convenient to take $P$ on the revolving line; that $PM$ is drawn perpendicular to the other line *produced if necessary;* and that the **order of the letters** in $MP$, $OP$, $OM$ is an essential part of the definition.

The **order of the letters** $P, M, O$ in the expressions $\frac{MP}{OP}$, etc., is therefore of great importance.

133. We proceed to show that the Trigonometrical Ratios of an angle vary in **Sign** according to the **Quadrant** in which the revolving line of the angle happens to be.

From the definition we have, with the usual letters,

$$\sin ROP = \frac{MP}{OP}, \cos ROP = \frac{OM}{OP}, \tan ROP = \frac{MP}{OM}.$$

Fig. I.

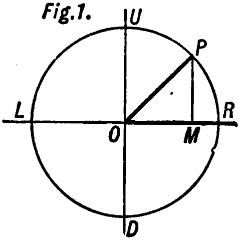


Fig. II.

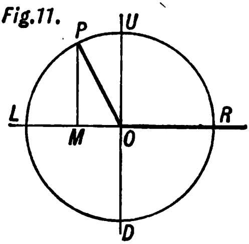


Fig. III.

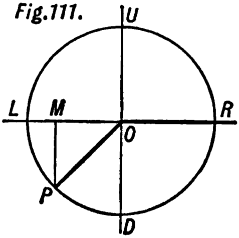
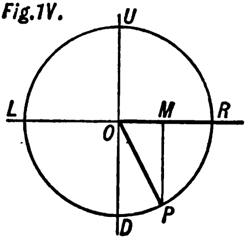


Fig. IV.



I. When $OP$ is in the first Quadrant (Fig. I.).

$MP$ is *positive* because from $M$ to $P$ is *upwards*
(Rule II. p. 95.)

$OM$ is *positive* because from $O$ to $M$ is *towards the right*
(Rule I.)

$OP$ is *positive* (Rule III.)

Hence, if $A$ be any angle of the *first* Quadrant,

$$\sin A, \text{ which is } \frac{MP}{OP}, \text{ is } \textit{positive};$$

$$\cos A, \text{ which is } \frac{OM}{OP}, \text{ is } \textit{positive};$$

$$\tan A, \text{ which is } \frac{MP}{OM}, \text{ is } \textit{positive}.$$

II. When $OP$ is in the **second** Quadrant, (Fig. II.).

$MP$ is *positive*, because from $M$ to $P$ is *upwards*,

$OM$ is *negative*, because from $O$ to $M$ is *towards the left.*

$OP$ is *positive.*

Hence, if $A$ be any angle of the *second* Quadrant,

$$\sin A, \text{ which is } \frac{MP}{OP}, \text{ is } \textit{positive};$$

$$\cos A, \text{ which is } \frac{OM}{OP}, \text{ is } \textit{negative};$$

$$\tan A, \text{ which is } \frac{MP}{OM}, \text{ is } \textit{negative}.$$

III. When $OP$ is in the **third** Quadrant (Fig. III.) $MP$ is *negative*, $OM$ is *negative*, $OP$ is *positive.*

So that, if $A$ be any angle of the *third* Quadrant, sin $A$ is *negative*, cos $A$ is *negative*, tan $A$ is *positive.*

IV. When $OP$ is in the **fourth** Quadrant (Fig. IV.) $MP$ is *negative*, $OM$ is *positive*, $OP$ is *positive.*

So that, if $A$ be any angle of the *fourth* Quadrant, sin $A$ is *negative*, cos $A$ is *positive*, tan $A$ is *negative.*

134. The table given below exhibits the results of the last article.

| Quadrant ... | I. | II. | III. | IV. |
|---|---|---|---|---|
| Sine ......... | + | + | − | − |
| Cosine ...... | + | − | − | + |
| Tangent ... | + | − | + | − |

The student should notice that in any particular Quadrant the three *signs* of sine, cosine, and tangent are unlike their signs in any other Quadrant.

## EXAMPLES. XXV.

State the *sign* of the sine, cosine, and tangent of each of the following angles:

(1) $60^0$. (2) $135^0$. (3) $265^0$.

(4) $275^0$. (5) $-10^0$. (6) $-91^0$.

(7) $-193^0$. (8) $-350^0$. (9) $-1000^0$.

(10) $2n\pi+\frac{\pi}{4}$. (11) $2n\pi+\frac{3\pi}{4}$. (12) $2n\pi-\frac{\pi}{6}$.

135. The NUMERICAL VALUES through which the Trigonometrical Ratios of the angle *ROP* pass, as *OP* moves through the *first* Quadrant, are **repeated** as *OP* moves through each of the other Quadrants.

Thus as *OP* moves through the second Quadrant from *U* to *L*, Fig. II, (*OP* being always of the same length) *MP* and *OM* pass through the same succession of numerical values through which they pass, as $P$ moves through the first Quadrant in the *opposite* direction from $U$ to $R$.

*Example* 1. *Find the sine, cosine and tangent of* $120^0$.

$120^0$ is an angle of the second Quadrant.

Let the angle $ROP$ be $120^0$ (Fig. II. p. 96.)

Then the angle $POL = 180^0 - 120^0 = 60^0$.

Hence, $\sin 120^0 = \frac{MP}{OP} = \sin 60^0$ *numerically*, and in the *second* Quadrant the sine is *positive*.

Therefore $$\sin 120^0 = \frac{\sqrt{3}}{2} \text{..........................(i).}$$

Again, $\cos 120^0 = \frac{OM}{OP} = \cos 60^0$, *numerically*, and in the *second* Quadrant the cosine is *negative*.

Therefore $$\cos 120^0 = -\frac{1}{2} \text{..........................(ii).}$$

Similarly, $$\tan 120^0 = -\sqrt{3} \text{.......................(iii).}$$

*Example* 2. *Find the sine, cosine and tangent of* $225^0$.

$225^0$ is an angle in the third Quadrant

Let the angle $ROP$ be $225^0$ (Fig. III.)

Here the angle $POL = 225^0 - 180^0 = 45^0$.

Therefore the Trigonometrical Ratios of $225^0 =$ those of $45^0$ *numerically*; and in the *third* Quadrant the sine and cosine are each *negative* and the tangent is *positive*.

Hence, $\sin 225^0 = -\frac{1}{\sqrt{2}}$; $\cos 225^0 = -\frac{1}{\sqrt{2}}$; $\tan 225^0 = 1$.

136. The cosecant, secant and cotangent of an angle $A$ have the same *sign* as the sine, cosine, and tangent of $A$ respectively.

## EXAMPLES. XXVI.

Find the algebraical value of the sine, cosine and tangent of the following angles :

(1) $150^0$. (2) $135^0$. (3) $-240^0$. (4) $330^0$.
(5) $-45^0$. (6) $-300^0$. (7) $225^0$. (8) $-135^0$.
(9) $390^0$. (10) $750^0$. (11) $-840^0$. (12) $1020^0$.
(13) $2n\pi+\frac{\pi}{4}$. (14) $(2n+1)\pi-\frac{\pi}{3}$. (15) $(2n-1)\pi+\frac{\pi}{6}$.

* *Example. To trace the changes in the magnitude and sign of sin* A, *as* A *increases from* $0^0$ *to* $360^0$.

Take the figure and construction of page 96.

As $A$ increases from $0^0$ to $90^0$, $MP$ increases from zero to $OP$, and is positive.

Therefore sin $A$ increases from 0 to 1 and is positive.

As $A$ increases from $90^0$ to $180^0$, $MP$ decreases from $OP$ to zero, and is positive.

Therefore sin $A$ decreases from 1 to 0 and is positive.

As $A$ increases from $180^0$ to $270^0$, $MP$ increases from zero to $OP$, and is negative.

Therefore sin $A$ increases numerically from 0 to 1 and is negative.

As $A$ increases from $270^0$ to $360^0$, $MP$ decreases from $OP$ to zero, and is negative.

Therefore sin $A$ decreases numerically from 1 to 0 and is negative.

## *EXAMPLES. XXVII.

Trace the changes in sign and magnitude as $A$ increases from $0^0$ to $360^0$ of

(1) $\cos A$. (2) $\tan A$. (3) $\cot A$. (4) $\sec A$.
(5) $\operatorname{cosec} A$. (6) $1-\sin A$. (7) $\sin^2 A$. (8) $\sin A.\cos A$.
(9) $\sin A+\cos A$. (10) $\tan A+\cot A$. (11) $\sin A-\cos A$.

## CHAPTER X**.

### ON ANGLES UNLIMITED IN MAGNITUDE.

137. JUST as the definitions of the Trigonometrical Ratios apply to angles of any magnitude whatever, so **every general Formula** involving these Ratios **is true for angles of any magnitude whatever.**

It is most important that the student should examine for himself into the truth of this statement.

138. The formulæ

$$\operatorname{cosec} A = \frac{1}{\sin A}, \quad \sec A = \frac{1}{\cos A}, \quad \cot A = \frac{1}{\tan A},$$

are really definitions; and since the definitions apply, therefore these formulæ are true, whatever be the magnitude of $A$.

The formulæ $\tan A = \dfrac{\sin A}{\cos A}$, $\cot A = \dfrac{\cos A}{\sin A}$,

are deduced immediately from the definitions, and therefore they are true whatever be the magnitude of $A$.

139. The formulæ $\sin^2 A + \cos^2 A = 1$,

$$1 + \tan^2 A = \sec^2 A,$$

$$1 + \cot^2 A = \operatorname{cosec}^2 A,$$

are each a trigonometrical statement of Euc. I. 47, and depend only on the fact that $MP$, $OM$, and $OP$ are the sides of a *right-angled* triangle. That this is the case, whatever be the magnitude of the angle $A$, is evident from the figures on page 96.

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* * To be omitted on a first reading, except pp. 104, 105.

140. In Art. 118 we proved that the sine of an angle is equal to the cosine of its complement, provided the angle lies between $0^\circ$ and $90^\circ$. We now give some examples of a method of proving the truth of this and other like formulæ, **whatever** be the magnitude of the angle concerned.

*Example* 1. *To prove that the sine of an angle = the cosine of its complement.*

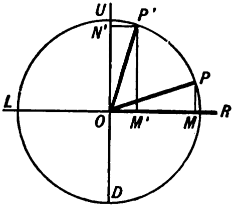
That is, *to prove* $\sin A = \cos(90^\circ - A)$

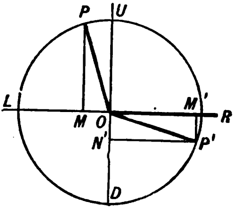
and $\cos A = \sin(90^\circ - A)$.

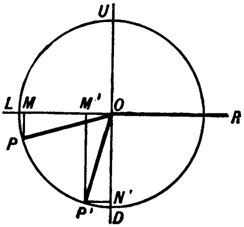
We take **two revolving lines** $OP$ and $OP'$. $OP$ starting from $OR$ is to describe the angle $A$; $OP'$ starting from $OR$ is to describe $(90^\circ - A)$.

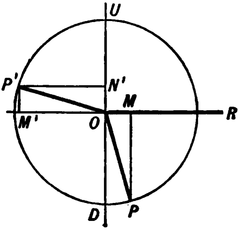
As usual, $PM$, $P'M'$ are perpendiculars on $OR$ and $P'N'$ is a perpendicular on $OU$.

In describing $(90^\circ - A)$ we shall consider that $OP'$ starting from $OR$, turns first through $90^\circ$ into the position $OU$, and then turns back from $OU$ through the angle $UOP' = (-A)$.









So that $ROP$, the angle which $OP$ describes from $OR$, is always equal to $UOP'$, the angle which $OP'$ describes from $OU$ in the *opposite* direction.

Hence, $N'P'$, that is $OM'$, is *always* equal to $MP$ in *magnitude.*

Also it will be seen that when $P$ is *above* $LOR$, $P'$ is to the *right* of $UOD$; when $P$ is *below* $LOR$, $P'$ is to the *left* of $UOD$.

Hence, $OM'$ and $MP$ have *always* the *same sign.*

Therefore $$\frac{MP}{OP}=\frac{OM'}{OP'} \text{ always,}$$

or, $$\sin A=\cos(90^0-A), \text{ for all values of } A.$$

Again, $ON'$, that is $M'P'$, is *always* equal to $OM$ in *magnitude.*

And $P'$ is above or below $LOR$ according as $P$ is to the right or to the left of $UOD$.

So that $M'P'$ and $OM$ have *always* the *same sign.*

Therefore $$\frac{OM}{OP}=\frac{M'P'}{OP'} \text{ always,}$$

or, $$\cos A=\sin(90^0-A) \text{ for all values of } A.$$

## EXAMPLES. XXVIII.

Prove, drawing a separate figure for each example, that

(1) $\sin 30^0=\cos 60^0$.  (2) $\sin 65^0=\cos 25^0$.

(3) $\sin 195^0=\cos(-105^0)$.  (4) $\cos 275=\sin(-185^0)$.

(5) $\cos(-27^0)=\sin 117^0$.  (6) $\cos 300^0=\sin(-210^0)$.

If $A$, $B$, $C$ be the angles of a triangle, so that $A+B+C=180^0$, prove

(7) $\cos\frac{A}{2}=\sin\frac{B+C}{2}$.  (8) $\cos\frac{B}{2}=\sin\frac{A+C}{2}$.

(9) $\sin\frac{C}{2}=\cos\frac{A+B}{2}$.  (10) $\sin\frac{A}{2}=\cos\frac{B+C}{2}$.

141. **Def.** Two angles are said to be the **Supplements** the one of the other when their sum is two right angles.

Thus $(180^0 - A)$ is the **supplement** of $A$

If $A$, $B$, $C$ be the angles of a triangle, $(A+B+C)=180^0$, so that $(B+C)$ is the **supplement** of $A$.

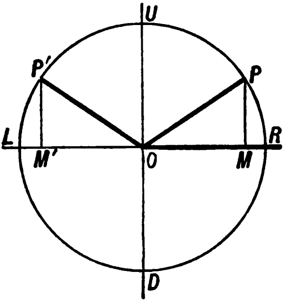
*Example* 2. *To prove that the sine of an angle = the sine of its supplement; and that the cosine of an angle* $= -$ *(the cosine of its supplement).*

That is, *to prove* $\sin A = \sin (180^0 - A)$

and $\cos A = -\cos (180^0 - A)$.

We take **two revolving lines** $OP$ and $OP'$. $OP$ starting from $OR$ describes the angle $A$; $OP'$ starting from $OR$ describes the angle $(180^0 - A)$.

In describing $(180^0 - A)$ we consider that $OP'$ starting from $OR$ turns first through $180^0$ into the position $OL$, and then back from $OL$ through the angle $LOP' = (-A)$.



So that $ROP$, the angle which $OP$ describes from $OR$, is always equal to $LOP'$, the angle which $OP'$ describes from $OL$ in the *opposite* direction.

Hence, $MP$ and $M'P'$ are *always* equal in *magnitude*.

Also, $P$ and $P'$ are always *both* above, or *both* below $LOR$.

So that $MP$ and $M'P'$ are *always* of the *same sign*.

Therefore $$\frac{MP}{OP} = \frac{M'P'}{OP'} \text{ always,}$$

or, $$\sin A = \sin(180^0 - A), \text{ for all values of } A.$$

Again, $OM$ and $OM'$ are *always* equal in *magnitude*.

Also it will be seen that when $P$ is on the *right* of $UOD$, $P'$ is on the *left* of $UOD$; when $P$ is on the *left* of $UOD$, $P'$ is on *right* of $UOD$.

So that $OM$ and $OM'$ are *always* of *opposite sign*.

Therefore $$\frac{OM}{OP} = -\frac{OM'}{OP'} \text{ always,}$$

or, $$\cos A = -\cos(180^0 - A), \text{ for all values of } A.$$

## EXAMPLES. XXIX.

Prove, drawing a separate figure in each case, that

(1) $\sin 60^0 = \sin 120^0$. (2) $\sin 340^0 = \sin(-160^0)$.

(3) $\sin(-40^0) = \sin 220^0$. (4) $\cos 320^0 = -\cos(-140^0)$

(5) $\cos(-380^0) = -\cos 560^0$. (6) $\cos 195^0 = -\cos(-15^0)$.

If $A$, $B$, $C$ be the angles of a triangle, prove

(1) $\sin A = \sin(B+C)$. (2) $\sin C = \sin(A+B)$.

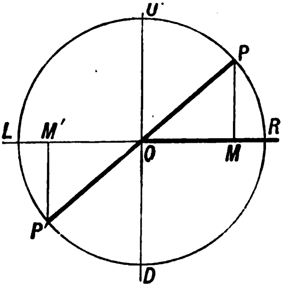
(3) $\cos B = -\cos(A+C)$ (4) $\cos A = -\cos(C+B)$.

*Example* 3. *To prove* $\sin A = -\sin(180^\circ + A)$,

and $\cos A = -\cos(180^\circ + A)$.

As before, we take two revolving lines $OP$ and $OP'$. $OP$ starting from $OR$ describes the angle $A$; $OP'$ starting from $OR$ describes the angle $(180^\circ + A)$.

In describing $(180^\circ + A)$ we consider that $OP'$ starting from $OR$ turns first through $180^\circ$ into the position $OL$, and then on from $OL$ through the angle $A$.



So that $ROP$, the angle which $OP$ describes from $OR$, is always equal to $LOP'$, the angle which $OP'$ describes from $OL$ in the *same* direction.

Hence, *$M'P'$ always* $= MP$ in *magnitude.*

Also it will be seen that when $P$ is *above* $LOR$, $P'$ is *below* $LOR$; and *vice versâ**.

So that $MP$ and $M'P'$ are *always* of *opposite sign*

Hence, $$\frac{MP}{OP} = -\frac{M'P'}{OP'} \text{ always,}$$

or, $$\sin A = -\sin(180^\circ + A), \text{ for all values of } A.$$

Similarly, *$OM$ always* $= OM'$ in *magnitude.*

And $P'$ is to the left or to the right of $UOD$ according as $P$ is to the right or to the left of $UOD$.

---

* This will be more clear if the student observes that $POP'$ is always a straight line.

Hence, $\frac{OM}{OP} = -\frac{OM'}{OP'}$ always,

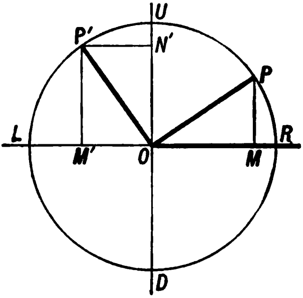
or, $\cos A = -\cos(180^\circ + A)$, for all values of $A$.

*Example* 4. *To prove* $\sin A = -\cos(90^\circ + A)$,

and $\cos A = \sin(90^\circ + A)$.

As before, we take two revolving lines $OP$ and $OP'$. $OP$ starting from $OR$ describes the angle $A$; $OP'$ starting from $OR$ describes the angle $90^\circ + A$.

In describing $(90^\circ + A)$ we consider that $OP'$ starting from $OR$ turns first through $90^\circ$ into the position $OU$, and then on from $OU$ through the angle $A$.



So that $ROP$, the angle which $OP$ describes from $OR$, is always equal to $UOP'$, the angle which $OP'$ describes from $OU$ in the *same* direction.

Hence, $N'P'$, that is $OM'$, *always* $= MP$ in *magnitude*.

Also, $P'$ is to the left or to the right of $UOD$ according as $P$ is above or below $LOR$.

So that $MP$ and $OM'$ are always of *opposite sign*.

Hence, $\frac{MP}{OP} = -\frac{OM'}{OP'}$ always,

or, $\sin A = -\cos(90^\circ + A)$, for all values of $A$.

Similarly, $OM$ *always* $= M'P'$ in *magnitude.*

And $P'$ is above or below $LOR$ according as $P$ is to the right or to the left of $UOD$.

So that $OM$ and $M'P'$ are *always* of the *same sign.*

Hence, $$\frac{OM}{OP} = \frac{M'P'}{OP'} \text{ always,}$$

or, $$\cos A = \sin(90^0 + A), \text{ for all values of } A.$$

## MISCELLANEOUS EXAMPLES. XXX.

Prove, drawing a separate figure in each case, that

(1) $\sin 60^0 = -\sin 240^0$. (2) $\sin 170^0 = -\sin 350^0$.

✓(3) $\sin(-20^0) = -\sin 160^0$. (4) $\cos 380^0 = -\cos 560^0$.

✓(5) $\cos(-225) = -\cos(-45^0)$. (6) $\cos 1005^0 = -\cos 1185^0$

✓(7) $\sin 60^0 = -\cos 150^0$. (8) $\cos 60^0 = \sin 150^0$

✓(9) $\sin 225 = -\cos 315^0$. (10) $\cos(-60^0) = \sin 30^0$.

If $A + B + C$ be the angles of a triangle, prove that

(11) $\sin A = -\sin(2A + B + C)$.

(12) $\sin A = -\cos \dfrac{3A + B + C}{2}$.

(13) $\cos B = \sin \dfrac{A + 3B + C}{2}$.

(14) $\cos C = -\cos(A + B + 2C)$.

(15) $\cos \dfrac{B - C}{2} = \sin \dfrac{A + 2B}{2}$.

(16) $\sin \dfrac{C - A}{2} = -\cos \dfrac{B + 2C}{2}$.

Prove the following statements for all values of $A$ and of $a$.

(17) $\sin A = -\sin(-A)$. (18) $\cos A = \cos(-A)$.

(19) $\sin A = \cos(A - 90^0)$. (20) $\cos A = -\sin(A - 90^0)$.

✓ (21) $\sin a = \cos\left(\frac{3\pi}{2} + a\right)$.

✓ (22) $\cos a = -\sin\left(\frac{3\pi}{2} + a\right)$

✓ (23) $\sin a = -\cos\left(\frac{3\pi}{2} - a\right)$.

✓ (24) $\cos a = -\sin\left(\frac{3\pi}{2} - a\right)$.

(25) $\sin\left(\frac{\pi}{2} - a\right) = \sin\left(\frac{\pi}{2} + a\right)$.

(26) $\cos(\pi + a) = \cos(\pi - a)$.

(27) $\tan(90^0 - A) = \cot A$. (28) $\tan A = -\tan(-A)$.

(29) $\tan(90^0 + A) = -\cot(A)$. (30) $\tan a = -\tan(\pi - a)$.

(31) $\tan A = \tan(180^0 + A)$. (32) $\cot\left(\frac{\pi}{2} - a\right) = \tan a$.

142. We have seen (Art. 135) that there are many angles of different magnitude which have the same *sine*.

If two such angles are in the *same Quadrant* they are represented geometrically by the same position of $OP$, so that they differ by some multiple of four right angles.

143. If we are given the value of the sine of an angle it is important to be able to find, Geometrically and Algebraically, all angles which have that value for their sine.

144. *To find the complete Geometrical Solution of the equation* $\sin\theta = a$.

With the usual construction, let the radius of the circle *RULD* be the unit of length; then the measure of *OP* is 1.

Fig. I.

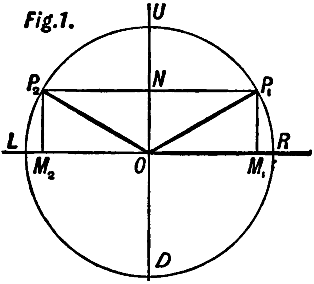
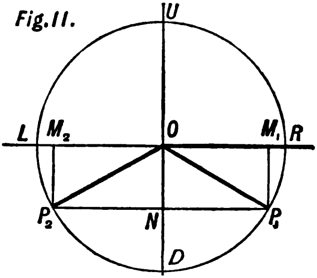


Fig. II.



From *O* draw on *OU* a line *ON* so that its measure is *a*.

[*ON* will be drawn upwards (Fig. I.) or downwards (Fig. II.) from *O* according as *a* is positive or negative.]

Through *N* draw $P_1P_2$ parallel to *LOR* to cut the circle in $P_1$ and $P_2$*. Join $OP_1$, $OP_2$. Draw $P_1M_1$, $P_2M_2$ perpendicular to *LOR*. Then $M_1P_1 = ON = M_2P_2$,

and $\frac{\text{perp.}}{\text{hyp.}}$ for $OP_1 = \frac{M_1P_1}{OP_1} = \frac{a}{1} = \frac{M_2P_2}{OP_2} = \frac{\text{perp.}}{\text{hyp.}}$ for $OP_2$.

Hence *every* angle whose initial line is *OR*, and whose final line is either $OP_1$ or $OP_2$, is an angle whose sine is *a*.

And no other angle has its sine equal to *a*, for there is no other possible position for *N*.

145. *To find an expression to include all angles having the given value* $a$ *for sine.*

With the usual construction, let the measure of the radius of the circle *RULD* be 1; on *OU* take *N* so that the measure of *ON* is *a*; through *N* draw $P_1NP_2$ parallel to

---

* If *a* were *greater than unity*, *ON* would be greater than *OU*, and the construction would fail.

$ROL$; join $OP_1$, $OP_2$. Then *every* angle whose initial line is $OR$ and final line, either $OP_1$, or $OP_2$, and *no other*, is an angle whose sine is $a$.

Let $ROP_1$ contain $a$ radians; then $ROP_2 = \pi - a$.

*Fig.* 1.

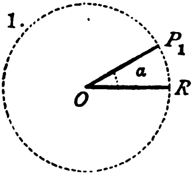
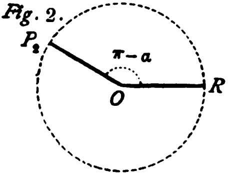


*Fig.* 2.



Every angle whose initial line is $OR$ and final line $OP_1$ is one of those included in the expression

$$2m\pi + a,$$

where $m$ is some integer, positive or negative. [Art. 129.]

Every angle whose initial line is $OR$ and final line $OP_2$ is one of those included in the expression

$$2r\pi + \pi - a, \text{ or } (2r+1)\pi - a,$$

where $r$ is some integer, positive or negative*. [Art. 129.]

Both of these expressions are included in

$$n\pi + (-1)^n a,$$

where $n$ is some integer, positive or negative*. This is therefore the required expression.

*Example.* Find six angles between $-4$ right angles and $+8$ right angles which satisfy the equation $\sin A^\circ = \sin 18^\circ$.

$$\text{We have } \left[\theta = n\pi + (-1)^n \frac{\pi}{10}, \text{ or}\right] A^\circ = n \times 180^\circ + (-1)^n 18^\circ.$$

Put for $n$ the values $-2$, $-1$, $0$, $1$, $2$, $3$, $4$ successively and we get the six angles

$$-360^\circ + 18^\circ,\ -180^\circ - 18^\circ,\ 18^\circ,\ 180^\circ - 18^\circ,\ 360^\circ + 18^\circ,\ 540^\circ - 18^\circ,$$

$$\text{i.e. } -342^\circ,\ -198^\circ,\ 18^\circ,\ 162^\circ,\ 378^\circ,\ 522^\circ.$$

---

[*] For if $n$ be *even*, let it be $2m$, when $(-1)^{2m} = +1$; if $n$ be *odd*, let it be $2r+1$, when $(-1)^{2r+1} = -1$.

The student is recommended to draw a figure in the above example. Also to draw a figure in each example of this kind which he works for exercise.

## EXAMPLES. XXXI.

(1) Find the four smallest angles which satisfy the equations

(i) $\sin A = \frac{1}{2}$. (ii) $\sin A = \frac{1}{\sqrt{2}}$. (iii) $\sin A = \frac{\sqrt{3}}{2}$. (iv) $\sin A = -\frac{1}{2}$.

(2) Find four angles between zero and $+8$ right angles which satisfy the equations

(i) $\sin A = \sin 20^0$. (ii) $\sin \theta = -\frac{1}{\sqrt{2}}$. (iii) $\sin \theta = -\sin \frac{\pi}{7}$.

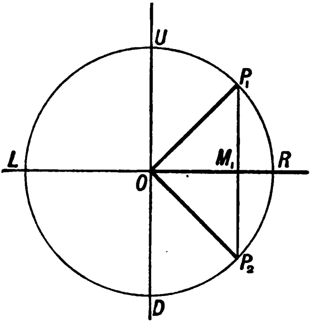
(3) Find the complete algebraical solution of

(i) $\sin \theta = -\frac{1}{2}$. (ii) $2 \sin^2 \theta + 3 \sin \theta = 2$. (iii) $\sin^2 \theta = \cos^2 \theta$.

(4) Prove that $30^0$, $150^0$, $-330^0$, $390^0$, $-210^0$ have all the same sine.

146. *To find the complete Geometrical Solution of the equation* $\cos \theta = a$.

With the usual construction, let the radius of the circle *RULD* be the unit of length, so that the measure of *OP* is 1.



From *O* draw on *OR* a line $OM_1$ such that its measure is $a$.

$OM_1$ will be drawn towards the right or towards the left according as $a$ is positive or negative.

Through $M_1$ draw $P_1M_1P_2$ perpendicular to $OR$ to cut the circle in $P_1$, $P_2$. Join $OP_1$, $OP_2$. Then,

$$\frac{\text{base}}{\text{hyp.}} \text{ for } OP_1 = \frac{OM_1}{OP_1} = \frac{a}{1} = \frac{OM_1}{OP_2} = \frac{\text{base}}{\text{hyp.}} \text{ for } OP_2.$$

Hence every angle whose initial line is $OR$, and final line either $OP_1$, or $OP_2$, is an angle whose **cosine** is $a$.

And no other angle has its **cosine** $= a$; for there is no other possible position of $M_1$.

147. *To find the complete Algebraical Solution of the equation* **cos** $\theta = a$.

With the usual construction, let the radius of the circle $RULD$ be 1; on $OM_1$ take $M_1$ so that the measure of $OM_1$ is $a$; through $M_1$ draw $P_1MP_2$ parallel to $UOD$; join $OP_1$, $OP_2$. Then *every* angle whose initial line is $OR$ and final line either $OP_1$, or $OP_2$, and *no other*, is an angle whose cosine is $a$.

Let $ROP_1$ contain $\alpha$ radians; then $ROP_2 = -\alpha$.

Every angle whose initial line is $OR$ and final line $OP_1$, is one of those included in the expression

$$2m\pi + \alpha. \qquad \text{[Art. 129.]}$$

Every angle whose initial line is $OR$ and final line $OP_2$, is one of those included in the expression

$$2r\pi - \alpha. \qquad \text{[Art. 129.]}$$

Both of these expressions are included in

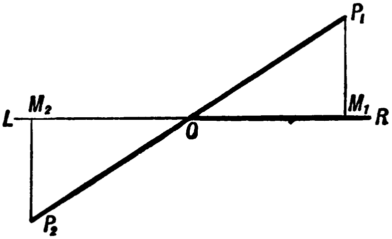
$$2n\pi \pm \alpha.$$

Thus the solution of the equation $\cos\theta = \cos\alpha$ is

$$\theta = 2n\pi \pm \alpha.$$

148. *To find the complete Geometrical Solution of the equation* **tan** $\theta = a$.

From $O$ draw on the line $OR$ two lines $OM_1$, $OM_2$, whose measures are $+1$ and $-1$ respectively.



And draw perpendicular to $LOR$ from $M_1$ a line $M_1P_1$ whose measure is $a$, and from $M_2$ a line $M_2P_2$ whose measure is $-a$. Join $OP_1$, $OP_2$. Then

$$\frac{\text{perp.}}{\text{base}} \text{ for } OP_1 = \frac{M_1P_1}{OM_1} = \frac{a}{1} = a,$$

and

$$\frac{\text{perp.}}{\text{base}} \text{ for } OP_2 = \frac{M_2P_2}{OM_2} = \frac{-a}{-1} = a.$$

Hence *every* angle whose initial line is $OR$, and final line either $OP_1$ or $OP_2$, is an angle whose **tangent** is $a$.

And no other angle has its **tangent** $= a$.

149. *To find the complete Algebraical Solution of the equation* **tan** $\theta = a$.

Let $OR$ be the initial line; from $O$ draw on $OR$ two lines $OM_1$, $OM_2$ whose measures are $+1$ and $-1$ respectively; from $M_1$, $M_2$ draw perpendicular to $LOR$ lines $M_1P_1$, $M_2P_2$ whose measures are $a$ and $-a$ respectively; join $OP_1$, $OP_2$. Then *every* angle whose initial line is $OR$ and final line either $OP_1$, or $OP_2$, and *no other*, is an angle whose tangent is $a$.

Let $ROP_1 = \alpha$; then, $ROP_2 = \pi + \alpha$.

Every angle whose initial line is $OR$ and final line $OP_1$, is one of those included in the expression

$$2m\pi + \alpha. \qquad \text{[Art. 129.]}$$

Every angle whose initial line is $OR$ and final line $OP_2$, is one of those included in the expression

$$2r\pi + (\pi + \alpha); \text{ or, } (2r + 1)\pi + \alpha. \qquad \text{[Art. 129.]}$$

Both of these expressions are included in

$$n\pi + \alpha^*.$$

Thus the solution of the equation $\tan\theta = \tan\alpha$ is

$$\theta = n\pi + \alpha.$$

## EXAMPLES. XXXII.

(1) Write down the complete Algebraical Solution of each of the following equations:

(i) $\cos\theta = \frac{1}{2}$. (ii) $\tan\theta = 1$. (iii) $\tan\theta = -1$.

(iv) $\tan\theta = -\sqrt{3}$. (v) $\cos\theta = \cos\frac{4\pi}{5}$. (vi) $\tan\theta = \tan\frac{3\pi}{4}$.

(2) Show that each of the following angles has the same cosine:

$$-120^0, \quad 240^0, \quad 480^0, \quad -480^0.$$

(3) The angles $60^0$ and $-120^0$ have one of the Trigonometrical Ratios the same for both; which of the ratios is it?

(4) Can the following angles have any one of their Trigonometrical Ratios the same for all?

$$-23^0, \quad -157^0 \text{ and } 157^0.$$

(5) Find four angles which satisfy each of the equations in (1).

---

[*] For if $n$ be even, this is the first formula; if $n$ be odd it is the second.

150. We can now point out the use of the ambiguous sign $\pm$ in the formula $\cos\theta = \pm\sqrt{1-\sin^2\theta}$.

If we know the numerical value of the sine of an angle $\theta$, without knowing the magnitude of the angle, we cannot from the identity, $\cos^2\theta = 1-\sin^2\theta$, *completely* determine $\cos\theta$, for we get $\cos\theta = \pm\sqrt{1-\sin^2\theta}$.

This is a general formula, and we shall find that it represents an important Geometrical truth.

151. Given $\sin\theta = a$, we can say that $\theta$ is *one of the angles* represented by one or the other of the positions $OP_1$, $OP_2$ of the revolving line in Fig. I. on page 110.

If we attempt to find the cosine of these angles we get two different values for the cosine; for $\frac{OM_1}{OP_1}$ and $\frac{OM_2}{OP_2}$ although equal in magnitude, are *opposite in sign*. Hence, if $\alpha$ be the least angle whose sine is equal to $a$, we have

$$\cos\theta = \pm\cos\alpha = \pm\sqrt{1-\sin^2\alpha}.$$

152. The same result may be obtained from the formula $\theta = n\pi + (-1)^n\alpha$. For $\cos\{n\pi + (-1)^n\alpha\}$ is of different sign according as $n$ is even or odd.

## EXAMPLES. XXXIII.

(1) If $\theta$ be found from the equation $\cos\theta = a$, show geometrically that there are two values of $\sin\theta$ and of $\tan\theta$.

(2) If $\theta$ be found from the equation $\tan\theta = a$, show geometrically that there are two values of $\sin\theta$ and of $\cos\theta$.

(3) If $A$ be the *least* angle without regard to sign such that $\sin A = a$, show that $\cos A = +\sqrt{1-\sin^2 A}$.

(4) If $A$ be the least *positive* angle such that $\cos A = a$, prove that $\sin A = +\sqrt{1-\cos^2 A}$.

The sum of sine of 2 ang = Sin 1st x Cos 2 + Co 1 x Sin 2

+

The sin diff 2 nos = sin 1st x sin 2 − [illegible]

The Cos sum two Nos = Cos 1 . Cos 2 − sin 1 . sin 2
" " diff two Nos = Cos 1 . Cos 2 + sin 1 . sin 2

## CHAPTER XI.

## ON THE TRIGONOMETRICAL RATIOS OF TWO ANGLES.

153. We proceed to establish the following fundamental formulæ:

Memorize

$$\left.\begin{aligned} \sin(A+B) &= \sin A \,.\, \cos B + \cos A \,.\, \sin B \\ \cos(A+B) &= \cos A \,.\, \cos B - \sin A \,.\, \sin B \\ \sin(A-B) &= \sin A \,.\, \cos B - \cos A \,.\, \sin B \\ \cos(A-B) &= \cos A \,.\, \cos B + \sin A \,.\, \sin B \end{aligned}\right\} \dots \text{(i)}.$$

Here, $A$ and $B$ are angles; so that $(A+B)$ and $(A-B)$ are also angles.

Hence, $\sin(A+B)$ is the sine of an angle, and must not be confounded with $\sin A + \sin B$.

$\text{Sin}(A+B)$ is a single fraction.

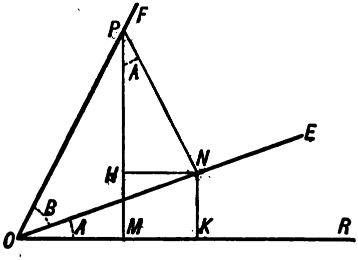
$\text{Sin}\, A + \sin B$ is the sum of two fractions.

154. The *proofs* given in the next two pages are perfectly general, as will be explained below (cf. Art. 169); but the *figures* are drawn for the simplest case in each.

The student should notice that the words of the two proofs are very nearly the same.

*To prove that* $\sin(A+B)=\sin A.\cos B+\cos A.\sin B$,

*and that* $\cos(A+B)=\cos A.\cos B-\sin A.\sin B$.



Let $ROE$ be the angle $A$, and $EOF$ the angle $B$. Then in the figure, $ROF$ is the angle $(A+B)$.

In $OF$, *the line which bounds the compound angle* $(A+B)$, take any point $P$, and from $P$ draw $PM$, $PN$ at right angles to $OR$ and $OE$ respectively. Draw $NH$, $NK$ at right angles to $MP$ and $OR$ respectively. Then the angle

$$NPH=90^0-HNP=HNO=ROE=A^*.$$

$$\text{Now } \sin(A+B)=\sin ROF=\frac{MP}{OP}=\frac{MH+HP}{OP}=\frac{KN}{OP}+\frac{HP}{OP}$$

$$=\frac{KN.\mathrm{ON}}{\mathrm{ON}.OP}+\frac{HP.\mathrm{NP}}{\mathrm{NP}.OP}=\frac{KN}{ON}\cdot\frac{ON}{OP}+\frac{HP}{NP}\cdot\frac{NP}{OP}$$

$$=\sin ROE.\cos EOF+\cos HPN.\sin EOF$$

$$=\sin A.\cos B+\cos A.\sin B.$$

$$\text{Also } \cos(A+B)=\cos ROF=\frac{OM}{OP}=\frac{OK-MK}{OP}=\frac{OK}{OP}-\frac{HN}{OP}$$

$$=\frac{OK.\mathrm{ON}}{\mathrm{ON}.OP}-\frac{HN.\mathrm{NP}}{\mathrm{NP}.OP}=\frac{OK}{ON}\cdot\frac{ON}{OP}-\frac{HN}{NP}\cdot\frac{NP}{OP}$$

$$=\cos ROE.\cos EOF-\sin HPN.\sin EOF$$

$$=\cos A.\cos B-\sin A.\sin B.$$

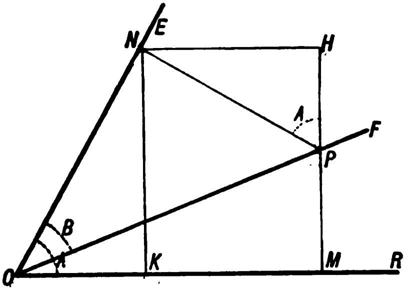
---

* Or thus. A circle goes round $OMNP$, because the angles $OMP$ and $ONP$ are right angles; therefore $MPN$ and $MON$ are angles in the same segment; so that $MPN=MON=A$.

[illegible]

*To prove that* $\sin(A-B)=\sin A.\cos B-\cos A.\sin B$,

*and that* $\cos(A-B)=\cos A.\cos B+\sin A.\sin B$.



Let $ROE$ be the angle $A$, and $FOE$ the angle $B$. Then in the figure, $ROF$ is the angle $(A-B)$.

In $OF$, *the line which bounds the compound angle* $(A-B)$, take any point $P$, and from $P$ draw $PM$, $PN$ at right angles to $OR$ and $OE$ respectively. Draw $NH$, $NK$ at right angles to $MP$ and $OR$ respectively. Then the angle

$$NPH=90^0-HNP=HNE=ROE=A^*.$$

Now $\sin(A-B)=\sin ROF=\frac{MP}{OP}=\frac{MH-PH}{OP}=\frac{KN}{OP}-\frac{PH}{OP}$

$$=\frac{KN.\mathrm{ON}}{\mathrm{ON}.OP}-\frac{PH.\mathrm{NP}}{\mathrm{NP}.OP}=\frac{KN}{ON}\cdot\frac{ON}{OP}-\frac{PH}{NP}\cdot\frac{NP}{OP}$$

$$=\sin ROE.\cos FOE-\cos HPN.\sin FOE$$

$$=\sin A.\cos B-\cos A.\sin B.$$

Also $\cos(A-B)=\cos ROF=\frac{OM}{OP}=\frac{OK+KM}{OP}=\frac{OK}{OP}+\frac{NH}{OP}$

$$=\frac{OK.\mathrm{ON}}{\mathrm{ON}.OP}+\frac{NH.\mathrm{NP}}{\mathrm{NP}.OP}=\frac{OK}{ON}\cdot\frac{ON}{OP}+\frac{NH}{NP}\cdot\frac{NP}{OP}$$

$$=\cos ROE.\cos FOE+\sin HPN.\sin FOE$$

$$=\cos A.\cos B+\sin A.\sin B.$$

---

* Or thus. A circle goes round $OMPN$, because the angles $OMP$ and $ONP$ are right angles; therefore $MPN$ and $MON$ together make up two right angles; so that $HPN=MON=A$.

*Example.* Find the value of $\sin 75^0$.

$$\begin{aligned}\sin 75^0 &= \sin(45^0+30^0)\\ &= \sin 45^0 . \cos 30^0 + \cos 45^0 . \sin 30^0\\ &= \frac{1}{\sqrt{2}} . \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} . \frac{1}{2}\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{2}(\sqrt{3}+1)}{4}.\end{aligned}$$

## EXAMPLES. XXXIV.

(1) Show that $\cos 75^0 = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

(2) Show that $\sin 15^0 = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

(3) Show that $\cos 15^0 = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

✓ (4) Show that $\tan 75^0 = 2+\sqrt{3}$.

✓ (5) If $\sin A = \frac{4}{5}$ and $\sin B = \frac{3}{5}$, find a value for $\sin(A+B)$ and for $\cos(A-B)$.

✓ (6) If $\sin A = \cdot 6$ and $\sin B = \frac{5}{13}$, find a value for $\sin(A-B)$ and for $\cos(A+B)$.

(7) If $\sin A = \dfrac{1}{\sqrt{5}}$ and $\sin B = \dfrac{1}{\sqrt{10}}$, show that one value of $(A+B)$ is $45^0$.

(8) Prove that $\sin 75^0 = \cdot 9659\ldots$

(9) Prove that $\sin 15^0 = \cdot 2588\ldots$

(10) Prove that $\tan 15^0 = \cdot 2679\ldots$

155. It is important that the student should become thoroughly familiar with the formulæ proved on the last two pages, and that he should be able to work examples involving their use.

## EXAMPLES. XXXV.

Prove the following statements.

(1) $\sin(A+B)+\sin(A-B)=2\sin A.\cos B.$

(2) $\sin(A+B)-\sin(A-B)=2\cos A.\sin B.$

(3) $\cos(A+B)+\cos(A-B)=2\cos A.\cos B.$

(4) $\cos(A-B)-\cos(A+B)=2\sin A.\sin B.$

(5) $\dfrac{\sin(A+B)+\sin(A-B)}{\cos(A+B)+\cos(A-B)}=\tan A.$

✓ (6) $\tan\alpha+\tan\beta=\dfrac{\sin(\alpha+\beta)}{\cos\alpha.\cos\beta}.$

✓ (7) $\tan\alpha-\tan\beta=\dfrac{\sin(\alpha-\beta)}{\cos\alpha.\cos\beta}.$

✓ (8) $\cot\alpha+\tan\beta=\dfrac{\cos(\alpha-\beta)}{\sin\alpha.\cos\beta}.$

✓ (9) $\cot\alpha-\tan\beta=\dfrac{\cos(\alpha+\beta)}{\sin\alpha.\cos\beta}.$

✓ (10) $\tan\alpha+\cot\beta=\dfrac{\cos(\alpha-\beta)}{\cos\alpha.\sin\beta}.$

(11) $\dfrac{\tan\theta+\tan\phi}{\tan\theta-\tan\phi}=\dfrac{\sin(\theta+\phi)}{\sin(\theta-\phi)}.$

(12) $\dfrac{\tan\theta.\tan\phi+1}{1-\tan\theta.\tan\phi}=\dfrac{\cos(\theta-\phi)}{\cos(\theta+\phi)}.$

(13) $\dfrac{\tan\theta+\cot\phi}{\cot\phi-\tan\theta}=\cos(\theta-\phi).\sec(\theta+\phi).$

(14) $\dfrac{\cot\theta+\cot\phi}{\cot\theta-\cot\phi}=-\dfrac{\sin(\theta+\phi)}{\sin(\theta-\phi)}.$

(15) $\dfrac{\tan\theta.\cot\phi+1}{\tan\theta.\cot\phi-1}=\dfrac{\sin(\theta+\phi)}{\sin(\theta-\phi)}.$

(16) $\dfrac{1+\cot\gamma.\tan\delta}{\cot\gamma-\tan\delta}=\tan(\gamma+\delta).$

(17) $\dfrac{1-\cot\gamma.\tan\delta}{\cot\gamma+\tan\delta}=\tan(\gamma-\delta).$

(18) $\dfrac{\tan\gamma \,.\, \cot\delta - 1}{\tan\gamma + \cot\delta} = \tan(\gamma - \delta).$

(19) $\dfrac{\tan\gamma \,.\, \cot\delta + 1}{\cot\delta - \tan\gamma} = \tan(\gamma + \delta).$

(20) $\dfrac{\cot\delta - \cot\gamma}{\cot\gamma \,.\, \cot\delta + 1} = \tan(\gamma - \delta).$

(21) $\tan^2 a - \tan^2\beta = \dfrac{\sin(a+\beta) \,.\, \sin(a-\beta)}{\cos^2 a \,.\, \cos^2\beta}.$

(22) $\cot^2 a - \tan^2\beta = \dfrac{\cos(a+\beta) \,.\, \cos(a-\beta)}{\sin^2 a \,.\, \cos^2\beta}.$

(23) $\dfrac{\tan^2 a - \tan^2\beta}{1 - \tan^2 a \,.\, \tan^2\beta} = \tan(a+\beta) \,.\, \tan(a-\beta).$

(24) $\sin(a+\beta) \,.\, \sin(a-\beta) = \sin^2 a - \sin^2\beta = \cos^2\beta - \cos^2 a.$

(25) $\cos(a+\beta) \,.\, \cos(a-\beta) = \cos^2 a - \sin^2\beta = \cos^2\beta - \sin^2 a.$

(26) $\sin(A - 45^0) = \dfrac{\sin A - \cos A}{\sqrt{2}}.$

(27) $\sqrt{2} \,.\, \sin(A + 45^0) = \sin A + \cos A.$

(28) $\cos A - \sin A = \sqrt{2} \,.\, \cos(A + 45^0).$

(29) $\cos(A + 45^0) + \sin(A - 45^0) = 0.$

(30) $\cos(A - 45^0) = \sin(A + 45^0).$

(31) $\sin(\theta + \phi) \,.\, \cos\theta - \cos(\theta + \phi) \,.\, \sin\theta = \sin\phi.$

(32) $\sin(\theta - \phi) \,.\, \cos\phi + \cos(\theta - \phi) \,.\, \sin\phi = \sin\theta.$

(33) $\cos(\theta + \phi) \,.\, \cos\theta + \sin(\theta + \phi) \,.\, \sin\theta = \cos\phi.$

(34) $\dfrac{\tan(\theta - \phi) + \tan\phi}{1 - \tan(\theta - \phi) \,.\, \tan\phi} = \tan\theta.$

(35) $\dfrac{\tan(\theta + \phi) - \tan\theta}{1 + \tan(\theta + \phi) \,.\, \tan\theta} = \tan\phi.$

(36) $2\sin\left(a + \dfrac{\pi}{4}\right) \,.\, \cos\left(\beta - \dfrac{\pi}{4}\right) = \cos(a - \beta) + \sin(a + \beta).$

(37) $2\sin\left(\dfrac{\pi}{4} - a\right) \,.\, \cos\left(\dfrac{\pi}{4} + \beta\right) = \cos(a - \beta) - \sin(a + \beta).$

(38) $\cos(a + \beta) + \sin(a - \beta) = 2\sin\left(\dfrac{\pi}{4} + a\right) \,.\, \cos\left(\dfrac{\pi}{4} + \beta\right).$

(39) $\cos(a + \beta) - \sin(a - \beta) = 2\sin\left(\dfrac{\pi}{4} - a\right) \,.\, \cos\left(\dfrac{\pi}{4} - \beta\right).$

tan sum two ∠s = tan 1st + tan 2nd

1 — prod of tans

(40) $\sin nA \,.\, \cos A + \cos nA \,.\, \sin A = \sin (n+1) A.$

(41) $\cos (n-1) A \,.\, \cos A - \sin (n-1) A \,.\, \sin A = \cos nA.$

(42) $\sin nA \,.\, \cos (n-1) A - \cos nA \,.\, \sin (n-1) A = \sin A.$

(43) $\cos (n-1) A \,.\, \cos (n+1) A - \sin (n-1) A \,.\, \sin (n+1) A = \cos 2nA.$

+ **156.** The following formulæ are important:

$$\left.\begin{aligned} \tan (A+B) &= \frac{\tan A + \tan B}{1 - \tan A \,.\, \tan B} \\ \tan (A-B) &= \frac{\tan A - \tan B}{1 + \tan A \,.\, \tan B} \end{aligned}\right\} \quad \ldots\ldots\ldots \text{(ii)}.$$

$$\tan (A + 45^\circ) = \frac{\tan A + 1}{1 - \tan A},$$

$$\tan (A - 45^\circ) = \frac{\tan A - 1}{1 + \tan A}.$$

The proof of the first is given below. The student should prove the second in a similar manner. The third and fourth follow at once from the first two by putting 45° for $B$.

*Example.* To prove $\tan (A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \,.\, \tan B}$.

(i) By using the results of Art. 154, we have

$$\tan (A+B) = \frac{\sin (A+B)}{\cos (A+B)} = \frac{\sin A \,.\, \cos B + \cos A \,.\, \sin B}{\cos A \,.\, \cos B - \sin A \,.\, \sin B}.$$

Divide the numerator and the denominator of this fraction each by $\cos A \,.\, \cos B$, and we get

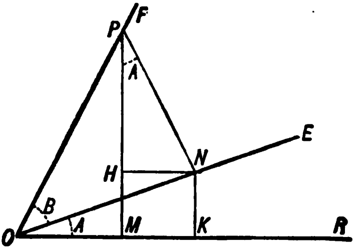
$$\tan (A+B) = \frac{\dfrac{\sin A \,.\, \cos B}{\cos A \,.\, \cos B} + \dfrac{\cos A \,.\, \sin B}{\cos A \,.\, \cos B}}{\dfrac{\cos A \,.\, \cos B}{\cos A \,.\, \cos B} - \dfrac{\sin A \,.\, \sin B}{\cos A \,.\, \cos B}}$$

$$= \frac{\tan A + \tan B}{1 - \tan A \,.\, \tan B}. \quad \text{Q.E.D.}$$

tan diff 2 ∠s tan 1st — tan 2nd

1 + prod of ∠s

(ii)* * By Geometry. Make the construction of page 118;



Then $\tan(A+B) = \tan ROF = \dfrac{MP}{OM} = \dfrac{MH+HP}{OK-MK}$

$$=\frac{\dfrac{KN}{OK}+\dfrac{HP}{OK}}{\dfrac{OK}{OK}-\dfrac{HN}{OK}}=\frac{\dfrac{KN}{OK}+\dfrac{HP}{OK}}{1-\dfrac{HN\,.\,\mathrm{HP}}{\mathrm{HP}\,.\,OK}}=\frac{\tan A+\dfrac{HP}{OK}}{1-\tan A\,.\,\dfrac{HP}{OK}}.$$

Now the triangles $NOK$ and $NPH$ are similar,

$$\therefore \frac{HP}{OK}=\frac{NP}{ON}=\tan B,$$

$$\therefore \tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\,.\,\tan B}. \qquad \text{Q.E.D.}$$

## EXAMPLES. XXXVI.

(1) If $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{4}$, prove that $\tan(A+B) = \frac{6}{7}$, and $\tan(A-B) = \frac{2}{9}$.

(2) If $\tan A = 1$ and $\tan B = \dfrac{1}{\sqrt{3}}$, prove that

$$\tan(A+B) = 2+\sqrt{3}.$$

(3) Prove that $\tan 15^0 = 2-\sqrt{3}$.

(4) If $\tan A = \frac{5}{6}$ and $\tan B = \frac{1}{11}$, prove that $\tan(A+B) = 1$. What is $(A+B)$ in this case?

(5) If $\tan A = m$ and $\tan B = \frac{1}{m}$, prove that $\tan(A+B) = \infty$. What is $(A+B)$ in this case?

Prove the following statements :

(6) $\cot(A+B) = \frac{\cot A \,.\, \cot B - 1}{\cot A + \cot B}$.

(7) $\cot(A-B) = \frac{\cot A \,.\, \cot B + 1}{\cot B - \cot A}$.

(8) $\cot\left(\theta - \frac{\pi}{4}\right) = \frac{\cot\theta + 1}{1 - \cot\theta}$.

(9) $\frac{\cot\theta - 1}{\cot\theta + 1} = \cot\left(\theta + \frac{\pi}{4}\right)$.

(10) $\tan\left(\theta - \frac{\pi}{4}\right) + \cot\left(\theta + \frac{\pi}{4}\right) = 0$.

(11) $\cot\left(\theta - \frac{\pi}{4}\right) + \tan\left(\theta + \frac{\pi}{4}\right) = 0$.

(12) If $\tan\alpha = \frac{m}{m+1}$ and $\tan\beta = \frac{1}{2m+1}$, prove that

$$\tan(\alpha+\beta) = 1.$$

(13) $\frac{\tan(n+1)\phi - \tan n\phi}{1 + \tan(n+1)\phi \,.\, \tan n\phi} = \tan\phi$.

(14) $\frac{\tan(n+1)\phi + \tan(1-n)\phi}{1 - \tan(n+1)\phi \,.\, \tan(1-n)\phi} = \tan 2\phi$.

(15) If $\tan\alpha = m$ and $\tan\beta = n$, prove that

$$\cos(\alpha+\beta) = \frac{1 - mn}{\sqrt{(1+m^2)(1+n^2)}}.$$

(16) If $\tan\alpha = (a+1)$ and $\tan\beta = (a-1)$, then $2\cot(\alpha-\beta) = a^2$.

(17) If $\alpha + \beta + \gamma = 90^0$, then

$$\tan\gamma = \frac{1 - \tan\alpha\tan\beta}{\tan\alpha + \tan\beta}.$$

157. From pages 118 and 119 we have

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$$\left.\begin{aligned}
\sin(A+B) &= \sin A\,.\cos B + \cos A\,.\sin B\\
\sin(A-B) &= \sin A\,.\cos B - \cos A\,.\sin B\\
\cos(A+B) &= \cos A\,.\cos B - \sin A\,.\sin B\\
\cos(A-B) &= \cos A\,.\cos B + \sin A\,.\sin B
\end{aligned}\right\}\ldots\text{(i).}$$

From these by addition and subtraction we get

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$$\left.\begin{aligned}
\sin(A+B) + \sin(A-B) &= 2\sin A\,.\cos B\\
\sin(A+B) - \sin(A-B) &= 2\cos A\,.\sin B\\
\cos(A+B) + \cos(A-B) &= 2\cos A\,.\cos B\\
\cos(A-B) - \cos(A+B) &= 2\sin A\,.\sin B
\end{aligned}\right\}$$

Now put $S$ for $(A+B)$,

and put $T$ for $(A-B)$.

Then $S+T=2A$, and $S-T=2B$,

so that $A=\dfrac{S+T}{2}$, and $B=\dfrac{S-T}{2}$.

Hence the above results may be written

$$\left.\begin{aligned}
\sin S + \sin T &= 2\sin\frac{S+T}{2}\,.\cos\frac{S-T}{2}\\
\sin S - \sin T &= 2\cos\frac{S+T}{2}\,.\sin\frac{S-T}{2}\\
\cos S + \cos T &= 2\cos\frac{S+T}{2}\,.\cos\frac{S-T}{2}\\
{}^{*}\cos T - \cos S &= 2\sin\frac{S+T}{2}\,.\sin\frac{S-T}{2}
\end{aligned}\right\}\ldots\text{(iii).}$$

---

[*] If $A$ and $B$ are each less than 90°, then $S$, which is their *sum*, is greater than $T$, their *difference*. Therefore if $S$ be less than 90°, $\cos S$ is less than $\cos T$; so that $\cos T - \cos S$ is *positive*.

+

158. The formulæ (iii) are most important, and the student is recommended to get thoroughly familiar with them *in words*, thus:

(1) *The sum of the sines of two angles is equal to twice the sine of half their sum multiplied by the cosine of half their difference.*

(2) *The sine of the greater of two angles minus the sine of the lesser is equal to twice the cosine of half their sum multiplied by the sine of half their difference.*

R (3) *The sum of the cosines of two angles is equal to twice the cosine of half their sum multiplied by the cosine of half their difference.*

(4) *The cosine of the lesser of two angles minus the cosine of the greater is equal to twice the sine of half their sum multiplied by the sine of half their difference.*

159. It will be convenient to refer to the formulæ (i) as the '*A,B*' formulæ, and to the formulæ (iii) as the '*S,T*' formulæ.

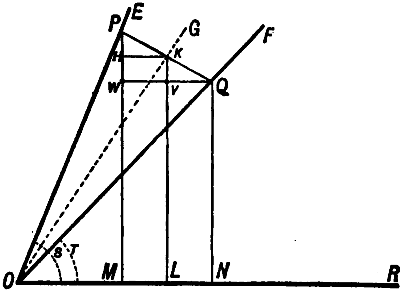
160. The '*S,T*' formulæ can be proved directly from a figure.

We give the proof for the case in which $S$ and $T$ are each less than 90°.

On first reading the subject however the student should omit the next two pages, and all other alternative proofs.

**(1) *To prove that*

$$\sin S + \sin T = 2 \sin \frac{S+T}{2} . \cos \frac{S-T}{2} .$$



In the figure, let $ROE$ be the angle $S$, $ROF$ the angle $T$; then $FOE$ is the angle $S-T$. Let $OG$ bisect the angle $EOF$.

Then the angle $FOG = \frac{FOE}{2} = \frac{S-T}{2}$.

Also the angle $ROG = ROF + FOG = T + \frac{S-T}{2} = \frac{S+T}{2}$.

In $OE$ take any point $P$, and from $OF$ cut off $OQ$ equal to $OP$. Join $PQ$, cutting $OG$ in $K$. Then $OG$ which bisects the vertical angle of the isosceles triangle $POQ$, bisects $PQ$ at right angles. Draw $PM$, $KL$, $QN$ at right angles to $OR$, and draw $QVW$ and $KH$ parallel to $NLM$.

Then by parallels, since $PK = KQ$, therefore $PH = HW$ and $WV = VQ$. Hence,

$$\sin S + \sin T = \frac{MP}{OP} + \frac{NQ}{OQ} = \frac{MP+NQ}{OQ}$$

$$= \frac{(LK+HP)+(LK-WH)}{OQ} = \frac{2LK}{OQ}$$

$$= \frac{2LK . \text{OK}}{\text{OK} . OQ} = 2 \sin ROK . \cos QOK$$

$$= 2 \sin \frac{S+T}{2} . \cos \frac{S-T}{2} . \quad \text{Q. E. D.}$$

(2) *To prove*

$$\cos S+\cos T=2\cos\frac{S+T}{2}\,.\,\cos\frac{S-T}{2}.$$

With the same figure and construction, we have

$$\cos S+\cos T=\frac{OM}{OP}+\frac{ON}{OQ}=\frac{OM+ON}{OQ},$$

$$=\frac{(OL-ML)+(OL+LN)}{OQ}=\frac{2OL}{OQ},$$

$$=\frac{2OL\,.\,\mathrm{OK}}{\mathrm{OK}\,.\,OQ}=2\cos ROK\,.\,\cos QOK,$$

$$=2\cos\frac{S+T}{2}\,.\,\cos\frac{S-T}{2}.\quad\text{Q.E.D.}$$

(3) *To prove*

$$\sin S-\sin T=2\sin\frac{S-T}{2}\,.\,\cos\frac{S+T}{2}.$$

In the above figure the angles $VKQ$, $ROK$ are each the complement of $LKO$ $\quad\therefore\; VKQ=ROK=\frac{S+T}{2}$. Hence

$$\sin S-\sin T=\frac{MP}{OP}-\frac{NQ}{OQ}=\frac{MP-NQ}{OQ},$$

$$=\frac{(LK+HP)-(LK-WH)}{OQ}=\frac{2VK}{OQ},$$

$$=\frac{2VK\,.\,\mathrm{KQ}}{\mathrm{KQ}\,.\,OQ}=2\cos VKQ\,.\,\sin QOK,$$

$$=2\sin\frac{S-T}{2}\,.\,\cos\frac{S+T}{2}.\quad\text{Q.E.D.}$$

(4) *To prove*

$$\cos T-\cos S=2\sin\frac{S+T}{2}\,.\,\sin\frac{S-T}{2}.$$

With the same figure and construction, we have

$$\cos T-\cos S=\frac{ON}{OQ}-\frac{OM}{OP}=\frac{ON-OM}{OQ},$$

$$=\frac{(OL+LN)-(OL-ML)}{OQ}=\frac{2LN}{OQ}.$$

$$=\frac{2VQ\,.\,\mathrm{KQ}}{\mathrm{KQ}\,.\,OQ}=2\sin VKQ\,.\,\sin QOK,$$

$$=2\sin\frac{S+T}{2}\,.\,\sin\frac{S-T}{2}.\quad\text{Q.E.D.}$$

## EXAMPLES. XXXVII.

Prove the following statements :

(1) $\sin 60^0 + \sin 30^0 = 2 \sin 45^0 . \cos 15^0.$

(2) $\sin 60^0 + \sin 20^0 = 2 \sin 40^0 . \cos 20^0.$

(3) $\sin 40^0 - \sin 10^0 = 2 \cos 25^0 . \sin 15^0.$

(4) $\cos \frac{\pi}{3} + \cos \frac{\pi}{2} = 2 \cos \frac{5\pi}{12} . \cos \frac{\pi}{12}.$

(5) $\cos \frac{\pi}{3} - \cos \frac{\pi}{2} = 2 \sin \frac{5\pi}{12} . \sin \frac{\pi}{12}.$

(6) $\sin 3A + \sin 5A = 2 \sin 4A . \cos A.$

(7) $\sin 7A - \sin 5A = 2 \cos 6A . \sin A.$

(8) $\cos 5A + \cos 9A = 2 \cos 7A . \cos 2A.$

(9) $\cos 5A - \cos 4A = -2 \sin \frac{9A}{2} . \sin \frac{A}{2}.$

(10) $\cos A - \cos 2A = 2 \sin \frac{3A}{2} . \sin \frac{A}{2}.$

(11) $\dfrac{\sin 2\theta + \sin \theta}{\cos \theta + \cos 2\theta} = \tan \dfrac{3\theta}{2}.$

(12) $\dfrac{\sin 2\theta - \sin \theta}{\cos \theta - \cos 2\theta} = \cot \dfrac{3\theta}{2}.$

(13) $\dfrac{\sin 3\theta + \sin 2\theta}{\cos 2\theta - \cos 3\theta} = \cot \dfrac{\theta}{2}.$

(14) $\dfrac{\sin \theta + \sin \phi}{\cos \theta - \cos \phi} = \dfrac{\cos \theta + \cos \phi}{\sin \phi - \sin \theta}.$

(15) $\cos (60^0 + A) + \cos (60^0 - A) = \cos A.$

(16) $\cos (45^0 + A) + \cos (45^0 - A) = \sqrt{2} . \cos A.$

(17) $\sin (45^0 + A) - \sin (45^0 - A) = \sqrt{2} . \sin A.$

(18) $\cos (30^0 - A) - \cos (30^0 + A) = \sin A.$

(19) $\dfrac{\sin \theta - \sin \phi}{\cos \phi - \cos \theta} = \cot \dfrac{\theta + \phi}{2}.$

(20) $\dfrac{\sin \theta - \sin \phi}{\sin \theta + \sin \phi} = \cot \left(\dfrac{\theta + \phi}{2}\right) . \tan \left(\dfrac{\theta - \phi}{2}\right).$

161. It is important that the student should be thoroughly familiar with the second set of formulæ on p. 126.

Written as follows, they may be regarded as the inverse of the '*S*, *T*' formulæ.

$$\left.\begin{aligned}2\sin A\,.\cos B &= \sin(A+B)+\sin(A-B),\\ 2\cos A\,.\sin B &= \sin(A+B)-\sin(A-B),\\ 2\cos A\,.\cos B &= \cos(A+B)+\cos(A-B),\\ 2\sin A\,.\sin B &= \cos(A-B)-\cos(A+B).\end{aligned}\right\}\text{ iv.}$$

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## EXAMPLES. XXXVIII.

Express as the sum or as the difference of two trigonometrical ratios the ten following expressions:

(1) $2\sin\theta\,.\cos\phi$.

(2) $2\cos\alpha\,.\cos\beta$.

(3) $2\sin 2\alpha\,.\cos 3\beta$.

(4) $2\cos(\alpha+\beta)\,.\cos(\alpha-\beta)$.

(5) $2\sin 3\theta\,.\cos 5\theta$.

(6) $2\cos\frac{3\theta}{2}\,.\cos\frac{\theta}{2}$.

(7) $\sin 4\theta\,.\sin\theta$.

(8) $\cos\frac{5\theta}{2}\,.\sin\frac{3\theta}{2}$.

(9) $2\cos 10^0\,.\sin 50^0$.

(10) $\cos 45^0\,.\sin 15^0$.

(11) Simplify $2\cos 2\theta\,.\cos\theta-2\sin 4\theta\,.\sin\theta$.

✓ (12) Simplify $\sin\frac{5\theta}{2}\,.\cos\frac{\theta}{2}-\sin\frac{9\theta}{2}\,.\cos\frac{3\theta}{2}$

(13) Simplify $\sin 3\theta+\sin 2\theta+2\sin\frac{3\theta}{2}\,.\cos\frac{\theta}{2}$.

(14) Prove that $\sin\frac{11\theta}{4}\,.\sin\frac{\theta}{4}+\sin\frac{7\theta}{4}\,.\sin\frac{3\theta}{4}=\sin 2\theta\,.\sin\theta$.

## * * MISCELLANEOUS EXAMPLES. XXXIX.

(1) If $\tan\alpha=\frac{1}{2}$ and $\tan\beta=\frac{1}{3}$, prove that $\tan(\alpha+\beta)=1$.

(2) If $\tan\alpha=\frac{3}{4}$ and $\tan\beta=\frac{1}{7}$, prove that one of the values of $\alpha+\beta$ is $\frac{\pi}{4}$.

(3) If $\tan\alpha=\frac{m-1}{m}$ and $\tan\beta=\frac{1}{2m-1}$, shew that one value of $(\alpha+\beta)$ is $\frac{\pi}{4}$.

(4) Simplify $\frac{\cos\alpha-\cos 5\alpha}{\sin\alpha+\sin 5\alpha}$.

(5) Simplify $\frac{\sin 5x-\sin 3x}{\cos 5x+\cos 3x}$.

(6) Prove that $\frac{\cos A+\cos 3A}{\cos 3A+\cos 5A}=\frac{\cos 2A}{\cos 4A}$.

(7) Simplify $\frac{\sin 3x-\sin x}{\cos 3x+\cos x}+\frac{\sin 3x+\sin x}{\cos 3x-\cos x}$.

(8) Simplify $\frac{(\sin 4A-\sin 2A)(\cos A-\cos 3A)}{(\cos 4A+\cos 2A)(\sin A+\sin 3A)}$.

(9) Prove that

$$2\sin 2\alpha\,.\,\cos\alpha+2\cos 4\alpha\,.\,\sin\alpha=\sin 5\alpha+\sin\alpha.$$

(10) Prove that

$$\cos 2\alpha\,.\,\cos\alpha-\sin 4\alpha\,.\,\sin\alpha=\cos 3\alpha\,.\,\cos 2\alpha.$$

(11) $\tan 2A\,.\,\tan 3A\,.\,\tan 5A=\tan 5A-\tan 3A-\tan 2A$.

(12) Solve $\left.\begin{array}{l}4\sin(\theta+\phi)\,.\,\cos(\theta-\phi)=3\\ 4\cos(\theta+\phi)\,.\,\sin(\theta-\phi)=1\end{array}\right\}$.

(13) Prove that

$$\frac{\sin A\,.\,\sin 2A+\sin 2A\,.\,\sin 5A+\sin 3A\,.\,\sin 10A}{\cos A\,.\,\sin 2A+\sin 2A\,.\,\cos 5A-\cos 3A\,.\,\sin 10A}=-\tan 7A.$$

(14) $\tan\frac{A+B}{2}-\tan\frac{A-B}{2}=\frac{2\sin B}{\cos A+\cos B}$.

## CHAPTER XII.

### ON THE TRIGONOMETRICAL RATIOS OF MULTIPLE ANGLES.

162. **To express the Trigonometrical Ratios of the angle $2A$ in terms of those of the angle $A$.**

Since $\sin(A+B) = \sin A \,.\, \cos B + \cos A \,.\, \sin B$;

$$\therefore\ \sin(A+A) = \sin A \,.\, \cos A + \cos A \,.\, \sin A;$$

$$\therefore\ \sin 2A = 2 \sin A \,.\, \cos A \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Also, since $\cos(A+B) = \cos A \,.\, \cos B - \sin A \,.\, \sin B$;

$$\therefore\ \cos(A+A) = \cos A \,.\, \cos A - \sin A \,.\, \sin A;$$

$$\therefore\ \cos 2A = \cos^2 A - \sin^2 A \quad \ldots\ldots\ldots\ldots \quad (2).$$

But $$1 = \cos^2 A + \sin^2 A;$$

$$\therefore\ 1 + \cos 2A = 2\cos^2 A,$$

and $$1 - \cos 2A = 2\sin^2 A.$$

The last two results are usually written

$$\cos 2A = 2\cos^2 A - 1 \quad \ldots\ldots\ldots\ldots\ldots \quad (3),$$

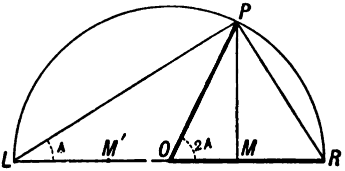
and $$\cos 2A = 1 - 2\sin^2 A \quad \ldots\ldots\ldots\ldots\ldots \quad (4).$$

Again, $$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \,.\, \tan B};$$

$$\therefore\ \tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A \,.\, \tan A};$$

$$\therefore\ \tan 2A = \frac{2\tan A}{1 - \tan^2 A} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5).$$

163.** *To prove the '2A' formulæ geometrically.*



Let $ROP$ be the angle $2A$. With centre $O$ and any radius describe the semicircle $RPL$. Draw $PM$ perpendicular to $OR$. Join $RP$ . $PL$.

Then the angle $RPL$ in a semicircle is a right angle. The angle $ROP=OLP+OPL=2\,OLP$ [since $OL=OP$]. $\therefore$ $OLP=$ a half of $ROP=A$. Also $MPR$ and $OLP$ are each the complement of $MPL$. $\therefore$ $MPR=OLP=A$.

Hence

(1) $$\sin 2A=\frac{MP}{OP}=\frac{2MP}{2OR}=\frac{2MP}{LR}=\frac{2MP\,.\,\mathrm{PR}}{\mathrm{PR}\,.\,LR}$$
$$=2\cos MPR\,.\,\sin PLR=2\sin A\,.\,\cos A.$$

(3) $$\cos 2A=\frac{OM}{OP}=\frac{LM-LO}{OP}=\frac{2LM}{2OP}-\frac{LO}{OP}$$
$$=\frac{2LM\,.\,\mathrm{LP}}{\mathrm{LP}\,.\,LR}-\frac{OP}{OP}=2\cos MLP\,.\,\cos PLR-1$$
$$=2\cos^2 A-1.$$

(2) Let $OM'=OM$.

Then $$2OM=M'M=LM-LM'=LM-MR.$$ Hence,
$$\cos 2A=\frac{2OM}{2OP}=\frac{LM-MR}{LR}=\frac{LM}{LR}-\frac{MR}{LR},$$
$$=\frac{LM\,.\,\mathrm{LP}}{\mathrm{LP}\,.\,LR}-\frac{MR\,.\,\mathrm{PR}}{\mathrm{PR}\,.\,LR}=\cos^2 A-\sin^2 A.$$

(5) $$\tan 2A=\frac{2MP}{2OM}=\frac{2MP}{LM-MR},$$
$$=\frac{\dfrac{2MP}{LM}}{\dfrac{LM}{LM}-\dfrac{MR}{LM}}=\frac{2\tan A}{1-\dfrac{MR\,.\,MP}{MP\,.\,LM}}=\frac{2\tan A}{1-\tan^2 A}.$$

164. These five formulæ are very important,

R

$$\left.\begin{array}{ll}\sin 2A = 2\sin A \,.\, \cos A & \ldots\ldots\ldots\ldots (1), \\ \left.\begin{array}{l}\cos 2A = \cos^2 A - \sin^2 A \\ \cos 2A = 2\cos^2 A - 1 \\ \cos 2A = 1 - 2\sin^2 A\end{array}\right\} & \begin{array}{l}\ldots\ldots\ldots (2), \\ \ldots\ldots\ldots (3), \\ \ldots\ldots\ldots (4),\end{array} \\ \tan 2A = \dfrac{2\tan A}{1-\tan^2 A} & \ldots\ldots\ldots\ldots (5),\end{array}\right\} \text{v.}$$

165. The following result is important,

$$\frac{\sin 2A}{1+\cos 2A} = \frac{2\sin A \,.\, \cos A}{2\cos^2 A} = \tan A.$$

166. The student must notice that $A$ is *any* angle, and therefore these formulæ will be true whatever we put for $A$.

*Example.* Write $\frac{A}{2}$ instead of $A$, and we get

$$\sin A = 2\sin\frac{A}{2} \,.\, \cos\frac{A}{2} \ldots\ldots\ldots\ldots (1),$$

$$\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} \ldots\ldots\ldots\ldots (2),$$

and so on.

## EXAMPLES. XL.

Prove the following statements :

(1) $2\operatorname{cosec} 2A = \sec A \,.\, \operatorname{cosec} A.$

(2) $\dfrac{\operatorname{cosec}^2 A}{\operatorname{cosec}^2 A - 2} = \sec 2A.$

(3) $\dfrac{2-\sec^2 A}{\sec^2 A} = \cos 2A.$

(4) $\cos^2 A\,(1-\tan^2 A) = \cos 2A$

(5) $\cot 2A = \dfrac{\cot^2 A - 1}{2\cot A}.$

(6) $\dfrac{2\tan B}{1+\tan^2 B}=\sin 2B.$

(7) $\tan B+\cot B=2\operatorname{cosec} 2B.$

(8) $\dfrac{1-\tan^2 B}{1+\tan^2 B}=\cos 2B.$

(9) $\cot B-\tan B=2\cot 2B.$

(10) $\dfrac{\cot^2 B+1}{\cot^2 B-1}=\sec 2B.$

(11) $\left(\sin\dfrac{\theta}{2}+\cos\dfrac{\theta}{2}\right)^2=1+\sin\theta.$

(12) $\left(\sin\dfrac{\theta}{2}-\cos\dfrac{\theta}{2}\right)^2=1-\sin\theta.$

(13) $\cos^2\dfrac{\theta}{2}\left(1+\tan\dfrac{\theta}{2}\right)^2=1+\sin\theta.$

(14) $\sin^2\dfrac{\theta}{2}\left(\cot\dfrac{\theta}{2}-1\right)^2=1-\sin\theta.$

(15) $\left(\dfrac{\tan\dfrac{\theta}{2}+1}{\tan\dfrac{\theta}{2}-1}\right)^2=\dfrac{1+\sin\theta}{1-\sin\theta}.$

(16) $\dfrac{\sin\beta}{1+\cos\beta}=\tan\dfrac{\beta}{2}.$

(17) $\dfrac{\sin\beta}{1-\cos\beta}=\cot\dfrac{\beta}{2}.$

(18) $\dfrac{1-\cos\beta}{1+\cos\beta}=\tan^2\dfrac{\beta}{2}.$

(19) $\dfrac{1+\sec\beta}{\sec\beta}=2\cos^2\dfrac{\beta}{2}.$

(20) $\operatorname{cosec}\beta-\cot\beta=\tan\dfrac{\beta}{2}.$

(21) $\dfrac{\cos 2x}{1+\sin 2x}=\dfrac{1-\tan x}{1+\tan x}.$

(22) $\dfrac{\cos x}{1-\sin x}=\dfrac{1+\tan\dfrac{x}{2}}{1-\tan\dfrac{x}{2}}.$

(23) $\dfrac{\cos x}{1+\sin x}=\dfrac{\cot\frac{x}{2}-1}{\cot\frac{x}{2}+1}$

(24) $\dfrac{\cos x}{1-\sin x}=\dfrac{\cot\frac{x}{2}+1}{\cot\frac{x}{2}-1}$.

(25) $\dfrac{1+\sin x+\cos x}{1+\sin x-\cos x}=\cot\dfrac{x}{2}$.

(26) $\dfrac{\cos^3 a+\sin^3 a}{\cos a+\sin a}=\dfrac{2-\sin 2a}{2}$.

(27) $\dfrac{\cos^3 a-\sin^3 a}{\cos a-\sin a}=\dfrac{2+\sin 2a}{2}$.

(28) $\cos^4 a-\sin^4 a=\cos 2a$.

(29) $\cos^6 a+\sin^6 a=\dfrac{1+3\cos^2 2a}{4}$.

(30) $\cos^6 a-\sin^6 a=\dfrac{(3+\cos^2 2a)\cos 2a}{4}$.

(31) $\dfrac{\sin 3\beta}{\sin\beta}-\dfrac{\cos 3\beta}{\cos\beta}=2$.

(32) $\dfrac{\cos 3\beta}{\sin\beta}+\dfrac{\sin 3\beta}{\cos\beta}=2\cot 2\beta$.

(33) $\dfrac{\sin 4\beta}{\sin 2\beta}=2\cos 2\beta$.

(34) $\dfrac{\sin 5\beta}{\sin\beta}-\dfrac{\cos 5\beta}{\cos\beta}=4\cos 2\beta$.

(35) $\dfrac{\sin\frac{5\pi}{12}}{\sin\frac{\pi}{12}}-\dfrac{\cos\frac{5\pi}{12}}{\cos\frac{\pi}{12}}=2\sqrt{3}$.

(36) $\tan(45^0+A)-\tan(45^0-A)=2\tan 2A$.

(37) $\tan(45^0-A)+\cot(45^0-A)=2\sec 2A$.

(38) $\dfrac{\tan^2(45^0+A)-1}{\tan^2(45^0+A)+1}=\sin 2A.$

(39) $\dfrac{\sec A+\tan A}{\sec A-\tan A}=\tan\left(45^0+\dfrac{A}{2}\right)\,.\,\cot\left(45^0-\dfrac{A}{2}\right).$

(40) $\dfrac{\cos(A+45^0)}{\cos(A-45^0)}=\sec 2A-\tan 2A.$

(41) $\tan B=\dfrac{\sin B+\sin 2B}{1+\cos B+\cos 2B}.$

(42) $\tan B=\dfrac{\sin 2B-\sin B}{1-\cos B+\cos 2B}.$

167. The following two formulæ should be remembered:

$$\left.\begin{aligned}\sin 3A&=3\sin A-4\sin^3 A\\ \cos 3A&=4\cos^3 A-3\cos A\end{aligned}\right\}\text{...............(vi).}$$

**Note.** The similarity of these two results is apt to cause confusion. This may be avoided by observing that the second formula must be true when $A=0^0$; and then $\cos 3A=\cos 0^0=1$. In which case the formula gives $\cos 0^0=4\cos 0^0-3\cos 0^0$, or $1=4-3$, which is true.

The first formula may be proved thus:

$$\begin{aligned}\sin 3A=\sin(2A+A)&=\sin 2A\,.\,\cos A+\cos 2A\,.\,\sin A\\ &=(2\sin A\,.\,\cos A)\cos A+(1-2\sin^2 A)\sin A\\ &=2\sin A\,.\,\cos^2 A+\sin A-2\sin^3 A\\ &=2\sin A(1-\sin^2 A)+\sin A-2\sin^3 A\\ &=2\sin A-2\sin^3 A+\sin A-2\sin^3 A\\ &=3\sin A-4\sin^3 A.\end{aligned}$$

The second formula may be proved in a similar manner.

*Example.* *Prove that* $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$.

$$\tan 3A = \tan(2A + A) = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$$

$$= \frac{\dfrac{2\tan A}{1-\tan^2 A} + \tan A}{1 - \dfrac{2\tan A}{1-\tan^2 A} \times \tan A}$$

$$= \frac{2\tan A + \tan A(1 - \tan^2 A)}{1 - \tan^2 A - 2\tan^2 A}$$

$$= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

## EXAMPLES. XLI.

Prove the following statements :

(1) $\dfrac{\sin 3A}{\sin A} = 2\cos 2A + 1.$

(2) $\dfrac{\cos 3A}{\cos A} = 2\cos 2A - 1.$

(3) $\dfrac{3\sin A - \sin 3A}{\cos 3A + 3\cos A} = \tan^3 A.$

(4) $\cot 3A = \dfrac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}.$

(5) $\dfrac{\sin 3A - \sin A}{\cos 3A + \cos A} = \tan A.$

(6) $\dfrac{\sin 3A - \cos 3A}{\sin A + \cos A} = 2\sin 2A - 1.$

(7) $\dfrac{\sin 3A + \cos 3A}{\cos A - \sin A} = 2\sin 2A + 1.$

(8) $\dfrac{1}{\tan 3A - \tan A} + \dfrac{1}{\cot A - \cot 3A} = \cot 2A.$

(9) $\left(\dfrac{3\sin A - \sin 3A}{3\cos A + \cos 3A}\right)^2 = \left(\dfrac{\sec 2A - 1}{\sec 2A + 1}\right)^3.$

(10) $\dfrac{1 - \cos 3A}{1 - \cos A} = (1 + 2\cos A)^2.$

## **MISCELLANEOUS EXAMPLES. XLII.

Prove the following statements :

(1) $\dfrac{\sin A+\cos A}{\cos A-\sin A}=\tan 2A+\sec 2A.$

(2) $\dfrac{\tan\dfrac{A}{2}+1}{1-\tan\dfrac{A}{2}}=\tan A+\sec A.$

(3) $\sin(n+1)a.\cos(n-1)a-\sin 2a=\sin(n-1)a.\cos(n+1)a.$

(4) $\dfrac{\sin a+\sin\beta}{\cos a+\cos\beta}=\tan\dfrac{a+\beta}{2}.$

(5) $\dfrac{\cos 2a+\cos 12a}{\cos 6a+\cos 8a}+\dfrac{\cos 7a-\cos 3a}{\cos a-\cos 3a}+2\dfrac{\sin 4a}{\sin 2a}=0.$

(6) If $A=18^0$, prove that $\sin 2A=\cos 3A$ ; hence prove that

$$\sin 18^0=\frac{\sqrt{5}-1}{4}.$$

(7) $\dfrac{\sin a+\sin\beta+\sin(a+\beta)}{\sin a+\sin\beta-\sin(a+\beta)}=\cot\dfrac{a}{2}.\cot\dfrac{\beta}{2}.$

(8) $\sin 2A.\sin 2B=\sin^2(A+B)-\sin^2(A-B).$

(9) $\cos 4A=8\cos^4 A-8\cos^2 A+1.$

(10) $\tan 50^0+\cot 50^0=2\sec 10^0.$

(11) $\sin 3A=4\sin A.\sin(60^0+A)\sin(60^0-A).$

(12) $\left(\cot\dfrac{A}{2}-\tan\dfrac{A}{2}\right)^2(\cot A-2\cot 2A)=4\cot A.$

(13) $\dfrac{\cos 3a-\sin\beta.\sin 5a-\cos 7a}{\sin 3a+\sin\beta.\cos 5a-\sin 7a}$ is independent of $\beta$.

(14) $(\cos x+\cos y)^2+(\sin x+\sin y)^2=4\cos^2\dfrac{x-y}{2}.$

(15) $2\cos^2 A.\cos^2 B+2\sin^2 A.\sin^2 B=1+\cos 2A.\cos 2B.$

(16) $\cot\dfrac{\pi}{8}-\tan\dfrac{\pi}{8}=2.$

(17) $\tan 4\theta=\dfrac{4\tan\theta(1-\tan^2\theta)}{1-6\tan^2\theta+\tan^4\theta}.$

(18) $2\cos\frac{\pi}{8}=\sqrt{2+\sqrt{2}}.$

(19) $2\cos 11^\circ\, 15'=\sqrt{2+\sqrt{2+\sqrt{2}}}.$

(20) $\dfrac{\sin A\,.\,\sin 2A+\sin A\,.\,\sin 4A+\sin 2A\,.\,\sin 7A}{\sin A\,.\,\cos 2A+\sin 2A\,.\,\cos 5A+\sin A\,.\,\cos 8A}=\tan 5A.$

(21) $\dfrac{\sin\theta+\sin(\theta+\phi)+\sin(\theta+2\phi)}{\cos\theta+\cos(\theta+\phi)+\cos(\theta+2\phi)}=\tan(\theta+\phi).$

(22) $2\cos^8 A-2\sin^8 A=\cos 2A\,(1+\cos^2 2A).$

(23) $(3\sin A-4\sin^3 A)^2+(4\cos^3 A-3\cos A)^2=1.$

(24) $\dfrac{\sin 2\alpha\,.\,\cos\alpha}{(1+\cos 2\alpha)\,(1+\cos\alpha)}=\tan\frac{\alpha}{2}.$

(25) $2\,\dfrac{\cot(n-2)\alpha\,.\,\cot n\alpha+1}{\cot(n-2)\alpha-\cot n\alpha}=\cot\alpha-\tan\alpha.$

(26) If $\tan\alpha=\frac{1}{7}$ and $\tan\beta=\frac{2}{11}$, prove $\tan(2\alpha+\beta)=\frac{1}{2}$.

(27) Prove that $\tan\frac{A}{2}$ and $\cot\frac{A}{2}$ are the roots of the equation

$$x^2-2x\,.\,\mathrm{cosec}\,A+1=0.$$

(28) If $\tan B=\frac{b}{a}$, prove that $\sqrt{\dfrac{a+b}{a-b}}+\sqrt{\dfrac{a-b}{a+b}}=\dfrac{2\cos B}{\sqrt{\cos 2B}}.$

*168. The following examples are symmetrical, and each involve more than two angles:

*Example* 1. Prove that

$$\sin(\alpha+\beta+\gamma)=\sin\alpha\,.\,\cos\beta\,.\,\cos\gamma+\sin\beta\,.\,\cos\gamma\,.\,\cos\alpha$$
$$+\sin\gamma\,.\,\cos\alpha\,.\,\cos\beta-\sin\alpha\,.\,\sin\beta\,.\,\sin\gamma.$$

$$\begin{aligned}\sin(\alpha+\beta+\gamma)&=\sin(\alpha+\beta)\,.\,\cos\gamma+\cos(\alpha+\beta)\sin\gamma\\&=\sin\alpha\,.\,\cos\beta\,.\,\cos\gamma+\cos\alpha\,.\,\sin\beta\,.\,\cos\gamma\\&\qquad+\cos\alpha\,.\,\cos\beta\,.\,\sin\gamma-\sin\alpha\,.\,\sin\beta\,.\,\sin\gamma\\&=\sin\alpha\,.\,\cos\beta\,.\,\cos\gamma+\sin\beta\,.\,\cos\gamma\,.\,\cos\alpha\\&\qquad+\sin\gamma\,.\,\cos\alpha\,.\,\cos\beta-\sin\alpha\,.\,\sin\beta\,.\,\sin\gamma.\end{aligned}$$

Q.E.D.

*Example* 2. Prove that

$$\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)$$
$$=4\,.\sin\frac{\beta+\gamma}{2}\,.\sin\frac{\gamma+\alpha}{2}\,.\sin\frac{\alpha+\beta}{2}.$$

Now $\sin\alpha-\sin(\alpha+\beta+\gamma)=-2\cos.\frac{2\alpha+\beta+\gamma}{2}\,.\sin\frac{\beta+\gamma}{2}$.

And $\sin\beta+\sin\gamma=2\sin\frac{\beta+\gamma}{2}\,.\cos\frac{\beta-\gamma}{2}$, [Art. 158]

$\therefore$ $\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)$

$$=2\sin\frac{\beta+\gamma}{2}\,.\cos\frac{\beta-\gamma}{2}-2\cos\frac{2\alpha+\beta+\gamma}{2}\,.\sin\frac{\beta+\gamma}{2}$$
$$=2\sin\frac{\beta+\gamma}{2}\left\{\cos\frac{\beta-\gamma}{2}-\cos\frac{2\alpha+\beta+\gamma}{2}\right\}$$
$$=2\sin\frac{\beta+\gamma}{2}\,.\,2\sin\frac{\gamma+\alpha}{2}\,.\sin\frac{\alpha+\beta}{2} \qquad \text{[Art. 158]}$$
$$=4\sin\frac{\beta+\gamma}{2}\,.\sin\frac{\gamma+\alpha}{2}\,.\sin\frac{\alpha+\beta}{2}. \qquad \text{Q.E.D.}$$

## * EXAMPLES. XLIII.

Prove the following statements;

(1) $\cos(\alpha+\beta+\gamma)=\cos\alpha\,.\cos\beta\,.\cos\gamma-\cos\alpha\,.\sin\beta\,.\sin\gamma$
$-\cos\beta\,.\sin\gamma\,.\sin\alpha-\cos\gamma\,.\sin\alpha\,.\sin\beta.$

(2) $\sin(\alpha+\beta-\gamma)=\sin\alpha\,.\cos\beta\,.\cos\gamma+\sin\beta\,.\cos\gamma\,.\cos\alpha$
$-\sin\gamma\,.\cos\alpha\,.\cos\beta+\sin\alpha\sin\beta\sin\gamma.$

(3) $\cos(\alpha-\beta+\gamma)=\cos\alpha\,.\cos\beta\,.\cos\gamma+\cos\alpha\,.\sin\beta\,.\sin\gamma$
$-\cos\beta\,.\sin\alpha\,.\sin\gamma+\cos\gamma\,.\sin\beta\,.\sin\alpha.$

(4) $\sin\alpha+\sin\beta-\sin\gamma-\sin(\alpha+\beta-\gamma)$
$=4\sin\frac{\alpha-\gamma}{2}\,.\sin\frac{\beta-\gamma}{2}\,.\sin\frac{\alpha+\beta}{2}.$

(5) $\sin(\alpha-\beta-\gamma)-\sin\alpha+\sin\beta+\sin\gamma$
$=4\sin\frac{\alpha-\beta}{2}\,.\sin\frac{\alpha-\gamma}{2}\,.\sin\frac{\beta+\gamma}{2}.$

(6) $\sin 2\alpha+\sin 2\beta+\sin 2\gamma-\sin 2(\alpha+\beta+\gamma)$
$=4\sin(\beta+\gamma)\,.\sin(\gamma+\alpha)\,.\sin(\alpha+\beta).$

(7) $$\sin(\beta-\gamma)+\sin(\gamma-\alpha)+\sin(\alpha-\beta) + 4\sin\frac{(\beta-\gamma)}{2}\,.\,\sin\frac{(\gamma-\alpha)}{2}\,.\,\sin\frac{(\alpha-\beta)}{2}=0.$$

(8) $$\sin(\beta+\gamma-\alpha)+\sin(\gamma+\alpha-\beta)+\sin(\alpha+\beta-\gamma) - \sin(\alpha+\beta+\gamma)=4\sin\alpha\,.\,\sin\beta\,.\,\sin\gamma.$$

(9) $$\sin(\alpha+\beta+\gamma)+\sin(\beta+\gamma-\alpha)+\sin(\gamma+\alpha-\beta) - \sin(\alpha+\beta-\gamma)=4\cos\alpha\,.\,\cos\beta\,.\,\sin\gamma.$$

(10) $$\cos x+\cos y+\cos z+\cos(x+y+z) =4\cos\frac{y+z}{2}\,.\,\cos\frac{z+x}{2}\,.\,\cos\frac{x+y}{2}.$$

(11) $$\cos 2x+\cos 2y+\cos 2z+\cos 2(x+y+z) =4\cos(y+z)\,.\,\cos(z+x)\,.\,\cos(x+y).$$

(12) $$\cos(y+z-x)+\cos(z+x-y)+\cos(x+y-z) +\cos(x+y+z)=4\cos x\,.\,\cos y\,.\,\cos z.$$

(13) $$\cos^2 x+\cos^2 y+\cos^2 z+\cos^2(x+y+z) =2\{1+\cos(y+z)\,.\,\cos(z+x)\,.\,\cos(x+y)\}.$$

(14) $$\sin^2 x+\sin^2 y+\sin^2 z+\sin^2(x+y+z) =2\{1-\cos(y+z)\,.\,\cos(z+x)\,.\,\cos(x+y)\}.$$

(15) $$\cos^2 x+\cos^2 y+\cos^2 z+\cos^2(x+y-z) =2\{1+\cos(x-z)\,.\,\cos(y-z)\,.\,\cos(x+y)\}.$$

(16) $$\cos\alpha\,.\,\sin(\beta-\gamma)+\cos\beta\,.\,\sin(\gamma-\alpha)+\cos\gamma\,.\,\sin(\alpha-\beta)=0.$$

(17) $$\sin\alpha\,.\,\sin(\beta-\gamma)+\sin\beta\,.\,\sin(\gamma-\alpha)+\sin\gamma\,.\,\sin(\alpha-\beta)=0.$$

(18) $$\cos(\alpha+\beta)\,.\,\cos(\alpha-\beta)+\sin(\beta+\gamma)\sin(\beta-\gamma) -\cos(\alpha+\gamma)\,.\,\cos(\alpha-\gamma)=0.$$

(19) $$\cos(\delta-\alpha)\,.\,\sin(\beta-\gamma)+\cos(\delta-\beta)\,.\,\sin(\gamma-\alpha) -\cos(\delta-\gamma)\,.\,\sin(\beta-\alpha)=0.$$

(20) $$8\cos\frac{\theta+\phi+\chi}{2}\,.\,\cos\frac{\phi+\chi-\theta}{2}\,.\,\cos\frac{\chi+\theta-\phi}{2}\,.\,\cos\frac{\theta+\phi-\chi}{2} =\cos 2\theta+\cos 2\phi+\cos 2\chi+4\cos\theta\,.\,\cos\phi\,.\,\cos\chi+1.$$

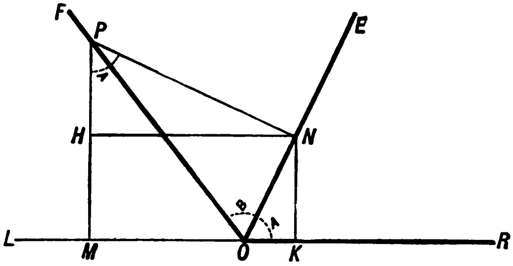
# CHAPTER XIII.**

## ON ANGLES UNLIMITED IN MAGNITUDE. II.

169. THE **words** of the proofs (on pages 118, 119) of the '$A$, $B$' formulæ apply to angles of any magnitude. The **figures** will be different for angles of different magnitude.

170. The figure for the '$A - B$' formulæ on page 119 is the same for all cases in which $A$ and $B$ are each less than 90°.

The figure given below is for the proof of the '$A + B$' formulæ, when, $A$ and $B$ being each less than 90°, their sum is greater than 90°.



The **words** of the proof are precisely those of page 118.

We may notice however that

$$\cos(A+B) = \frac{OM}{OP}\left[=\frac{-MO}{OP}\right] = \frac{-MK+OK}{OP} = \frac{OK}{OP} - \frac{MK}{OP},$$

and the rest follows as on page 118.

171. Thus we have proved that the '*A*, *B*' formulæ are true provided *A* and *B* each lie between 0° and 90°.

The student can prove them for any other values by drawing the proper figure.

The '*A*, *B*' formulæ are therefore true for any values whatever of the angles *A* and *B*.

172. By the aid of the '*A*, *B*' formulæ we can prove the formulæ of Art. 140.

*Example. Prove that* $\sin(90^0+A)=\cos A$.

$$\begin{aligned}\sin(90^0+A)&=\sin 90^0\cos A+\cos 90^0\sin A,\\&=1\times\cos A+0\times\sin A,\\&=\cos A.\quad \text{Q. E. D.}\end{aligned}$$

## EXAMPLES. XLIV.

Draw the figures for the first four of the following examples.

(1) For the $(A+B)$ formulæ, when $A$ is greater than $90^0$ and $(A+B)$ less than $180^0$.

(2) For the $(A-B)$ formulæ, when $A$ and $B$ each lie between $90^0$ and $180^0$.

(3) For the $(A+B)$ formulæ, when $A$ lies between $90^0$ and $180^0$, and $(A+B)$ lies between $180^0$ and $270^0$.

(4) For the $(A-B)$ formulæ, when $A$ lies between $180^0$ and $270^0$, and $(A-B)$ lies between $180^0$ and $A$.

Deduce the six following formulæ from the '*A*, *B*' formulæ.

(5) $\cos(90^0+A)=-\sin A$. (6) $\sin(90^0-A)=\cos A$.

(7) $\cos(90^0-A)=\sin A$. (8) $\sin(180^0-A)=\sin A$.

(9) $\cos(180^0-A)=-\cos A$. (10) $\sin(180^0+A)=-\sin A$.

(11) Assuming that the formula $\sin(A+B)=\sin A.\cos B+\cos A.\sin B$ is true for all values of $A$ and $B$, deduce the rest of the '*A*, *B*' formulæ by the aid of the results on p. 107.

173. We may also **conversely** prove that the '$A$, $B$' formulæ are true for angles of any magnitude by the aid of the result of page 107. (This is a very convenient method of proving the '$A$, $B$' formulæ to be true for all values of the angles.)

For, assuming that the '$A$, $B$' formulæ are true for certain values of the angles $A$ and $B$, we can show that they are true if either of the angles $A$ or $B$ be *increased* by 90°.

*Example.*

$$\begin{aligned}\sin(90^\circ+A+B)&=\cos(A+B) \qquad \text{[p. 107.]}\\ &=\cos A\,.\,\cos B-\sin A\,.\,\sin B,\\ &=\sin(90^\circ+A)\,.\,\cos B-\{-\cos(90^\circ+A)\}\sin B,\\ &=\sin(90^\circ+A)\,.\,\cos B+\cos(90^\circ+A)\,.\,\sin B.\end{aligned}$$

Or, writing $A'$ for $90^\circ+A$, we have

$$\sin(A'+B)=\sin A'\,.\,\cos B+\cos A'\,.\,\sin B.$$

We have proved (Art. 170) that the '$A$, $B$' formulæ are true for all values of $A$ and $B$ between 0° and 90°. And therefore, by what we have said above, they are true for all values of $A$ or $B$ between 0° and 180°. And so on.

Therefore the '$A$, $B$' formulæ are true for any values whatever of the angles $A$ and $B$.

174. It follows that all formulæ deduced from the '$A$, $B$' formulæ are true for angles of any magnitude whatever.

Thus the '$S$, $T$' formulæ (page 126) are true for angles of any magnitude. Also the formulæ of the last Chapter for multiple angles, and all general formulæ in the Examples, are true for angles of any magnitude.

## EXAMPLES. XLV.

(1) Deduce the values of sin $180^0$ and cos $180^0$ from those of the sine and cosine of $90^0$.

(2) The angle $A$ is greater than $180^0$ and less than $270^0$ and $\tan A = \frac{1}{2}$: find $\sin 2A$ and $\sin 3A$.

(3) The angle $\theta$ lies in the fourth quadrant and $\cos\theta = \frac{1}{2}$, find $\sin 2\theta$ and $\sin 3\theta$. Find also $\cos 3\theta$, and hence determine in which quadrant $3\theta$ lies.

(4) Prove that the different values of $\theta$ which satisfy the equation $\cos p\theta + \cos q\theta = 0$, form two series in *A. P.* with common differences $\dfrac{2\pi}{p+q}$ and $\dfrac{2\pi}{p \sim q}$ respectively.

### ON SUBMULTIPLE ANGLES.

175. We have now proved all the formulæ of the last two Chapters to be universally true.

We may expect therefore that any result, which can be obtained from these formulæ by algebraical transformation, will have a complete geometrical interpretation. [See p. 116.]

176. Since, $\cos A = 1 - 2\sin^2\dfrac{A}{2}$, [Art. 166.]

and, $\cos A = 2\cos^2\dfrac{A}{2} - 1$;

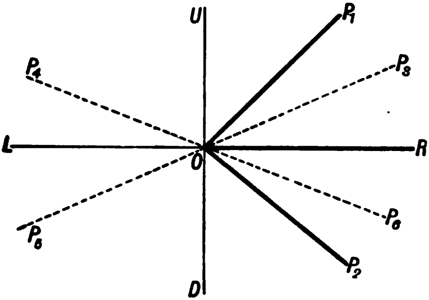
we have, $\sin^2\dfrac{A}{2} = \dfrac{1-\cos A}{2}$,

and, $\cos^2\dfrac{A}{2} = \dfrac{1+\cos A}{2}$.

Or,
$$\left.\begin{aligned} \sin\frac{A}{2} &= \pm\sqrt{\frac{1-\cos A}{2}} \\ \cos\frac{A}{2} &= \pm\sqrt{\frac{1+\cos A}{2}} \end{aligned}\right\}.$$

Thus, given the value of $\cos A$ (nothing else being known about the angle $A$), we get **two** values for $\sin\frac{A}{2}$, one positive and one negative, and **two** like values for $\cos\frac{A}{2}$.

177. *To prove geometrically that, given the value of* $\cos A$ (nothing else being known about the angle $A$), *there are* **two** *values each of* $\sin\frac{A}{2}$ *and of* $\cos\frac{A}{2}$.



Let $\alpha$ be the least positive angle which has the given cosine, and let $ROP_1$ and $P_2OR$ in the figure each $=\alpha$. Then $A$ is one of the angles described by the revolving line $OP$ when, starting from $OR$, $OP$ stops either in the position $OP$ or in the position $OP_2$; i.e. any one of the angles

$$2n\pi \pm \alpha. \qquad \text{[Art. 147]}$$

The angle described is (some multiple of *four* right angles $\pm a$); a half of this is (some multiple of *two* right angles $\pm \frac{1}{2}a$); so that a half of the angle whose cosine is equal to that of $a$, may be any one of the angles indicated by $OR$ and the four dotted lines $OP_3$ and $OP_5$, or $OP_4$ and $OP_6$ in the figure.

And it will be seen that

$$\sin ROP_3 = \sin ROP_4 = -\sin ROP_5 = -\sin ROP_6$$

Also, $\cos ROP_3 = -\cos ROP_4 = -\cos ROP_5 = \cos ROP_6$.

From these it is clear that there are **two** values of $\sin\frac{A}{2}$, equal in magnitude and opposite in sign. Also, that there are **two** like values for $\cos\frac{A}{2}$.

## EXAMPLES. XLVI.

(1) When $A$ lies between $-180^0$ and $180^0$, prove that

$$\cos\frac{A}{2} = +\sqrt{\frac{1+\cos A}{2}}.$$

(2) When $A$ lies between $180^0$ and $540^0$, prove that

$$\cos\frac{A}{2} = -\sqrt{\frac{1+\cos A}{2}}.$$

(3) Find $\sin\frac{A}{2}$ in terms of $\cos A$, when $A$ lies between $180^0$ and $360^0$.

(4) Prove that when $A$ lies between $(4n+1)\pi$ and $(4n+3)\pi$, $n$ being a positive integer, $\cos\frac{A}{2} = -\sqrt{\frac{1+\cos A}{2}}$.

(5) Find $\sin\frac{A}{2}$ in terms of $\cos A$, when $A$ lies between $4n\pi$ and $(4n+2)\pi$, where $n$ is a positive integer.

178. Since, $2\sin\frac{A}{2}\cdot\cos\frac{A}{2}=\sin A$, [Art. 166.]

and, $$\sin^2\frac{A}{2}+\cos^2\frac{A}{2}=1,$$

we obtain by addition and subtraction

$$\sin^2\frac{A}{2}+\cos^2\frac{A}{2}+2\sin\frac{A}{2}\cdot\cos\frac{A}{2}=1+\sin A,$$

$$\sin^2\frac{A}{2}+\cos^2\frac{A}{2}-2\sin\frac{A}{2}\cdot\cos\frac{A}{2}=1-\sin A.$$

Therefore $$\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)^2=1+\sin A,$$

and $$\left(\sin\frac{A}{2}-\cos\frac{A}{2}\right)^2=1-\sin A.$$

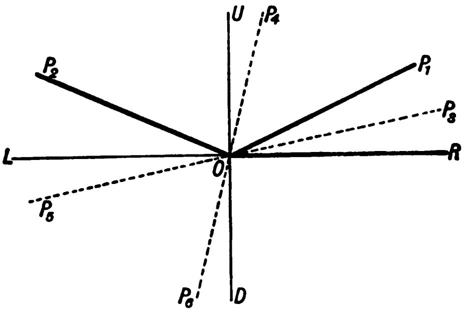
Whence, $$\left.\begin{array}{l}\sin\frac{A}{2}+\cos\frac{A}{2}=\pm\sqrt{1+\sin A}\\ \sin\frac{A}{2}-\cos\frac{A}{2}=\pm\sqrt{1-\sin A}\end{array}\right\}\begin{array}{l}\ldots\ldots(\text{i}),\\ \ldots\ldots(\text{ii}).\end{array}$$

Adding we get, Subtracting we get, $$\left.\begin{array}{l}2\sin\frac{A}{2}=\pm\sqrt{1+\sin A}\pm\sqrt{1-\sin A}\\ 2\cos\frac{A}{2}=\pm\sqrt{1+\sin A}\mp\sqrt{1-\sin A}\end{array}\right\}\begin{array}{l}(\text{iii}),\\ (\text{iv}).\end{array}$$

Thus if we are given the value of $\sin A$, (nothing else being known about the angle $A$), we have **four** values for $\sin\frac{A}{2}$ and **four** values for $\cos\frac{A}{2}$.

179. *To prove Geometrically that, given the value of* $\sin A$ (nothing else being known about the angle $A$), *there are* **four** *values for* $\sin\frac{A}{2}$, *and* **four** *values for* $\cos\frac{A}{2}$.

Let $\alpha$ be the least positive angle which has the given sine, and let $ROP_1$, $P_2OL$ in the figure each $=\alpha$.



Then $A$ is one of the angles described by the revolving line $OP$, when, starting from $OR$, $OP$ stops either in the position $OP_1$ or in the position $OP_2$.

The angle described is either (an *even* multiple of *two* right angles $+ a$) or (an *odd* multiple of *two* right angles $- a$); a half of this is either (an *even* multiple of one right angle $+ \frac{1}{2} a$) or (an *odd* multiple of one right angle $- \frac{1}{2} a$); so that a half of the angle whose sine is equal to that of $a$ may be any one of the angles indicated by $OR$ and the dotted lines $OP_3$ and $OP_5$, or $OP_4$ and $OP_6$ in the figure.

In the figure, $ROP_3 = P_4OU = LOP_5 = P_6OD$.

And $\sin\frac{A}{2}$ may have any one of the values $\sin ROP_3$, $\sin ROP_4$, $\sin ROP_5$, $\sin ROP_6$. These values are all different, and are those given by the solution (iii).

Hence, there are **four** values for $\sin\frac{A}{2}$, of the nature indicated by the solution (iii); also **four** values for $\cos\frac{A}{2}$ (iv).

180. If we know the magnitude of $A$, we can decide which sign to take in the formulæ

$$\sin\frac{A}{2}+\cos\frac{A}{2}=\pm\sqrt{1+\sin A}\ldots\ldots\ldots\ldots\text{(i)},$$

$$\sin\frac{A}{2}-\cos\frac{A}{2}=\pm\sqrt{1-\sin A}\ldots\ \ldots\ldots\ldots\text{(ii)}.$$

*Example.* When $\frac{A}{2}$ lies between $-45^0$ and $+45^0$,

$\cos\frac{A}{2}$ is *greater* than $\sin\frac{A}{2}$ and is *positive.*

So that $\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)$ is *positive*, and $=+\sqrt{1+\sin A}$,

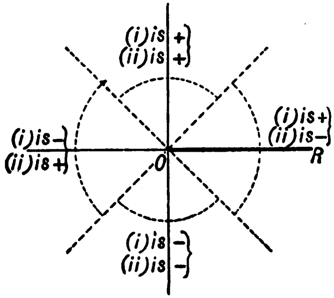
and $\left(\sin\frac{A}{2}-\cos\frac{A}{2}\right)$ is *negative*, and $=-\sqrt{1-\sin A}$.

When $\frac{A}{2}$ lies between $+45^0$ and $+135^0$,

$\sin\frac{A}{2}$ is greater than $\cos\frac{A}{2}$, and is positive.

So that $\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)$ and $\left(\sin\frac{A}{2}-\cos\frac{A}{2}\right)$ are both positive. And so on.

The following diagram completes the above results.



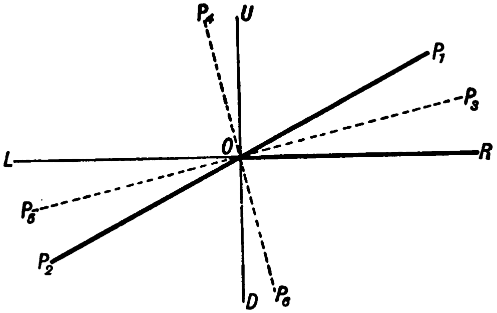
181. Since $$\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}},$$

$$\therefore \tan^2\frac{A}{2} + \frac{2\tan\frac{A}{2}}{\tan A} - 1 = 0,$$

whence $$\tan\frac{A}{2} = \frac{-1 \pm \sqrt{1+\tan^2 A}}{\tan A}.$$

Thus, given $\tan A$ we find two unequal values for $\tan\frac{A}{2}$ one positive and one negative.

182. The student will be able by the aid of the following figure to verify this result geometrically.



183. We may remark that in this figure $P_3OP_5$ and $P_4OP_6$ are straight lines at right angles to each other. So that $\tan P_3OR = -\cot P_4OR$; or, $\tan P_3OR \,.\, \tan P_4OR = -1$.

Hence, one value of $\tan\frac{A}{2}$ is the reciprocal of the other, and of opposite sign. So that there is always one *positive* value of $\tan\frac{A}{2}$, and one *negative;* one numerically *greater* than unity and the other numerically *less* than unity.

*Example.* *If* $A=190^\circ$, *prove that*

$$\tan\frac{A}{2}=\frac{-1-\sqrt{1+\tan^2 A}}{\tan A}.$$

Now $\tan A=\tan 190^\circ$, which is positive; $\tan\frac{A}{2}=\tan 95^\circ$, which is negative. Also $\tan\frac{A}{2}=\frac{-1\pm\sqrt{1+\tan^2 A}}{\tan A}$; the negative value of which is $\frac{-1-\sqrt{1+\tan^2 A}}{\tan A}$.

## EXAMPLES. XLVII.

(1) State the signs of $\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)$ and $\left(\sin\frac{A}{2}-\cos\frac{A}{2}\right)$ when $\frac{A}{2}$ has the following values:

(i) $22^\circ$, (ii) $191^\circ$, (iii) $290^\circ$, (iv) $345^\circ$,
(v) $-22^\circ$, (vi) $-275^\circ$, (vii) $-470^\circ$, (viii) $1000^\circ$.

(2) Prove that the formulæ which give the values of $\sin\frac{A}{2}$ and of $\cos\frac{A}{2}$ in terms of $\sin A$, are unaltered when $A$ has the values

(i) $92^\circ$, $268^\circ$, $900^\circ$, $4n\pi+\frac{3}{4}\pi$, or $(4n+2)\pi-\frac{3}{4}\pi$.

(ii) $88^\circ$, $-88^\circ$, $770^\circ$, $-770^\circ$, or $4n\pi\pm\frac{\pi}{8}$.

(3) Find the values of (i) $\sin 9^\circ$, (ii) $\cos 9^\circ$, (iii) $\sin 81^\circ$, (iv) $\cos 189^\circ$, (v) $\tan 202\frac{1}{2}^\circ$, (vi) $\tan 97\frac{1}{2}^\circ$.

(4) If $A=200^\circ$, prove that

(i) $2\sin\frac{A}{2}=+\sqrt{1+\sin A}+\sqrt{1-\sin A}$.

(ii) $\tan\frac{A}{2}=\frac{-(1+\sqrt{1+\tan^2 A})}{\tan A}$.

(5) If $A$ lie between $270^\circ$ and $360^\circ$, prove that

(i) $2\sin\frac{A}{2}=\sqrt{1-\sin A}-\sqrt{1+\sin A}$.

(ii) $\tan\frac{A}{2}=-\cot A+\operatorname{cosec} A$.

(6) If $A$ lie between $450^\circ$ and $630^\circ$ prove that

$$2\sin\frac{A}{2}=-\sqrt{1+\sin A}-\sqrt{1-\sin A}.$$

(7) Find the limits between which $\frac{A}{2}$ must lie when

$$2\sin\frac{A}{2}=\sqrt{1+\sin A}-\sqrt{1-\sin A},$$

(8) Given that $A$ lies between $450^0$ and $630^0$, prove that

$$2\cos\frac{A}{2}=-\sqrt{1+\sin A}+\sqrt{1-\sin A}.$$

(9) If $A$ lie between $n\times 360^0-90^0$ and $n\times 360^0+90^0$ where $n$ is a positive integer, prove that

$$\tan\frac{A}{2}=\frac{-1+\sqrt{1+\tan^2 A}}{\tan A};$$

and that when $A$ lies between $n\times 360^0+90^0$ and $n\times 360^0+270^0$,

then $$\tan\frac{A}{2}=\frac{-1-\sqrt{1+\tan^2 A}}{\tan A}.$$

(10) Prove geometrically that if we are given the value of $\sin A$, there are *three* different values for $\sin\frac{A}{3}$, and *six* different values for $\cos\frac{A}{3}$.

(11) Prove geometrically that, if we are given the value of $\cos A$, there are three different values for $\cos\frac{A}{3}$, and six different values for $\sin\frac{A}{3}$.

(12) Prove that if we are given the value of $\tan A$ there are three different values for $\tan\frac{A}{3}$.

(13) Given the value of $\tan A$, prove that there are *four* values each for $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$.

(14) Given the value of $\sin A$, prove that there are *two* values for $\tan\frac{A}{2}$

184. It is important if possible, in solving Trigonometrical equations, to avoid squaring both sides of the equation.

*Example.* *Solve* $\cos\theta = k\sin\theta$.

If we square both sides of the equation we get

$$\cos^2\theta = k^2\sin^2\theta = k^2(1-\cos^2\theta).$$

$$\therefore\ \cos^2\theta = \frac{k^2}{1+k^2},\ \text{or}\ \cos\theta = \pm\frac{k}{\sqrt{1+k^2}}.$$

Now if $\alpha$ be the least angle such that $\cos\theta = \frac{k}{\sqrt{1+k^2}}$,

then the above gives us $\theta = n\pi \pm \alpha$ ..................................(i).

But the equation may be written $\cot\theta = k$,

whence $\theta = n\pi + \alpha$ ..............................(ii).

(ii) is the complete solution of the proposed equation, while (i) is in fact the solution of both $\cos\theta = k\sin\theta$ and also of $\cos\theta = -k\sin\theta$. So that by squaring both sides of the equation we obtain solutions which do not belong to the given equation.

185. We can often avoid squaring by the use of a **Subsidiary** Angle.

*Example.* *Solve* $a\cos\theta + b\sin\theta = 1$.

That is $$a\left(\cos\theta + \frac{b}{a}\sin\theta\right) = 1.$$

Find in the tables the angle whose tangent is $\frac{b}{a}$; let it be $\alpha$.

Then $\frac{b}{a} = \tan\alpha$; and the equation becomes

$$a(\cos\theta + \tan\alpha\,.\,\sin\theta) = 1,$$

or $$a\left(\frac{\cos\theta\,.\,\cos\alpha + \sin\theta\,.\,\sin\alpha}{\cos\alpha}\right) = 1,$$

or $$\cos(\theta - \alpha) = \frac{\cos\alpha}{a}.$$

Find from the Tables the value of $\cos\alpha$. Next find from the Tables the magnitude of the angle $\beta$ whose cosine $=\dfrac{\cos\alpha}{a}$, and we get

$$\cos(\theta-\alpha)=\cos\beta;$$

$$\therefore\ \theta-\alpha=2n\pi\pm\beta,$$

or

$$\theta=\alpha+2n\pi\pm\beta.$$

186. **Def.** When an angle $\alpha$ is introduced to facilitate calculation it is called a **Subsidiary Angle.**

## EXAMPLES. XLVIII.

Solve the following equations.

(1) $2\sin\theta+2\cos\theta=\sqrt{2}$. (2) $\sin\theta+\sqrt{3}\,.\cos\theta=1$.

(3) $\sqrt{2}\sin\theta+\sqrt{2}\cos\theta=\sqrt{3}$. (4) $\sin\theta-\cos\theta=1$.

(5) $\sin\theta+\cos\theta=1$. (6) $\sqrt{3}\sin\theta-\cos\theta-\sqrt{2}=0$.

(7) $2\sin x+5\cos x=2$. $[2\cdot5=\tan 68^0\,12']$

(8) $3\cos x-8\sin x=3$. $[2\cdot\dot{6}=\tan 69^0\,26'\,30'']$

(9) $4\sin x-15\cos x=4$. $[3\cdot75=\tan 75^0\,4']$

(10) $\cos(\alpha+x)=\sin(\alpha+x)+\sqrt{2}\cos\beta$.

## *On the Inverse Notation.

187. The equation $\sin\theta=a$ means that $\theta$ is an angle whose sine is $a$.

$\theta=\sin^{-1}a$ is a convenient way of writing the same equation.

Thus $\sin^{-1}a$ (is an angle, and) is an abbreviation for **an angle whose sine is $a$.**

*Example* 1. *Show that* $30^0$ *is* one *value of* $\sin^{-1}\frac{1}{2}$.

We know that $\sin 30^0=\frac{1}{2}$. Therefore $30^0$ is an angle whose sine is $\frac{1}{2}$, or, $30^0=\sin^{-1}\frac{1}{2}$.

*Example* 2. *Prove that* $45^0$ *is one value of* $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$.

$\tan^{-1}\frac{1}{2}$ is one of the angles whose tangent is $\frac{1}{2}$.

Let $a=\tan^{-1}\frac{1}{2}$, so that, $\tan a=\frac{1}{2}$.

Let $\beta=\tan^{-1}\frac{1}{3}$, so that, $\tan\beta=\frac{1}{3}$.

We have to prove that $45^0$ is one value of $a+\beta$.

$$\tan(a+\beta)=\frac{\tan a+\tan\beta}{1-\tan a\,.\,\tan\beta}=\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\,.\,\frac{1}{3}}=1.$$

But $\tan 45^0=1$.

Therefore $45^0$ is one value of $a+\beta$, i.e. of $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$.

## * EXAMPLES. XLIX.

Prove that the following statements are true when we take for $\sin^{-1}a$, etc. their least *positive* value.

(1) $\sin^{-1}\frac{3}{5}=\cos^{-1}\frac{4}{5}=\tan^{-1}\frac{3}{4}$.

(2) $\sin^{-1}\frac{1}{2}=\cos^{-1}\frac{\sqrt{3}}{2}=\cot^{-1}\sqrt{3}$.

(3) $\sin^{-1}a=\cos^{-1}\sqrt{1-a^2}=\tan^{-1}\frac{a}{\sqrt{1-a^2}}$.

(4) If $a=\sin^{-1}\frac{3}{5}$ and $\beta=\cos^{-1}\frac{3}{5}$, then $a+\beta=\frac{\pi}{2}$.

(5) If $A=\sin^{-1}x$ and $B=\cos^{-1}a$, then $A+B=90^0$.

(6) $\tan^{-1}\frac{5}{7}+\tan^{-1}\frac{1}{6}=\frac{\pi}{4}$. (7) $\tan^{-1}\frac{2}{11}+2\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{1}{2}$.

(8) $\tan^{-1}m_1+\tan^{-1}m_2=\tan^{-1}\frac{m_1+m_2}{1-m_1m_2}$.

(9) $\sin(2\sin^{-1}a)=2a\sqrt{1-a^2}$.

(10) $2\cos^{-1}a=\cos^{-1}(2a^2-1)$.

(11) $\cos^{-1}\frac{1}{2}+2\sin^{-1}\frac{1}{2}=120^0$. (12) $2\sin^{-1}\frac{4}{5}-\sin^{-1}\frac{24}{25}=2\cos^{-1}\frac{24}{25}$.

(13) $2\tan^{-1}(\cos 2a)=\tan^{-1}\left(\frac{\cot^2 a-\tan^2 a}{2}\right)$.

(14) $\tan^{-1}x+\tan^{-1}y+\tan^{-1}\frac{1-x-y-xy}{1+x+y-xy}=\frac{\pi}{4}$.

(15) $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}=\frac{\pi}{4}$.

(16) $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{13}{85}=\frac{\pi}{2}$.

(17) $\tan^{-1}\sqrt{5}\,(2-\sqrt{3})-\cot^{-1}\sqrt{5}\,(2+\sqrt{3})=\cot^{-1}\sqrt{5}$.

(18) If $$\sin^{-1}m+\sin^{-1}n=\frac{\pi}{2},$$
prove that $$m\sqrt{1-n^2}+n\sqrt{1-m^2}=1.$$

188. The student must notice carefully that such a statement as $\sin^{-1}\frac{1}{2}=\cos^{-1}\frac{\sqrt{3}}{2}$ is not an identity.

For $\sin^{-1}\frac{1}{2}$ is one of the values of $n\pi+(-1)^n\,30°$,

and $\cos^{-1}\frac{\sqrt{3}}{2}$ is one of the values of $2n\pi\pm 30°$.

Thus $150°=\sin^{-1}\frac{1}{2}$, but $150°$ is not $=\cos^{-1}\frac{\sqrt{3}}{2}$.

## **MISCELLANEOUS EXAMPLES. L.

(1) Prove that $\tan^{-1}\frac{1}{1+a}+\tan^{-1}\frac{1}{1-a}+\tan^{-1}\frac{2}{a^2}=n\pi$.

(2) Prove that $\tan^{-1}\frac{a-1}{a}+\tan^{-1}\frac{1}{2a-1}=n\pi+\frac{\pi}{4}$.

(3) Prove that
$$\sin^{-1}x-\sin^{-1}y=\cos^{-1}\{xy\pm\sqrt{1-x^2-y^2+x^2y^2}\}.$$

(4) If $\tan^{-1}\frac{x+1}{x+2}+\tan^{-1}\frac{x-1}{x-2}=\frac{\pi}{4}$, show that $x^2=\frac{1}{2}$.

(5) Prove that $\tan^{-1}a+\cot^{-1}a=(2n+1)\frac{\pi}{2}$.

(6) If $$\tan^{-1}\alpha+\tan^{-1}\beta+\tan^{-1}\gamma=\pi,$$
prove that $$\alpha+\beta+\gamma=\alpha\beta\gamma.$$

(7) Solve the equation $\tan^{-1}\frac{1}{x-1}-\tan^{-1}\frac{1}{x+1}=\frac{\pi}{12}$.

✓ (8) Solve the equation

$$\tan^{-1}(x+1)-\tan^{-1}(x-1)=\cot^{-1}(x^2-1).$$

(9) Solve the equation $\sin^{-1}\dfrac{2x}{1+x^2}+\tan^{-1}\dfrac{2x}{1-x^2}=\dfrac{\pi}{2}$.

(10) Prove that

$$\tan^{-1}\frac{a-\sqrt{a^2-4}}{2\sqrt{a+1}}+\tan^{-1}\frac{1}{\sqrt{a+1}}+\tan^{-1}\frac{a+\sqrt{a^2-4}}{2\sqrt{a+1}}=n\pi+\frac{\pi}{2}$$

(11) If $a$ be positive and less than unity, and if $\alpha$ be the least value of $\sin^{-1}a$, then

$$\sin^{-1}a+\cos^{-1}a=n\pi+(-1)^n\alpha\pm\left(\frac{\pi}{2}-\alpha\right).$$

(12) Prove that $\tan A$ and $\sin 2A$ have always the same sign.

Solve the six following equations.

✓ (13) $\cos A+\cos 3A+\cos 5A=0.$

(14) $\sin 5\theta+\sin 3\theta+\sin\theta=3-4\sin^2\theta.$

(15) $2\sin^2 3A+\sin^2 6A=2.$

(16) $a(\cos 2x-1)+2b(\cos x+1)=0.$

(17) $\sin(m+n)\theta+\sin 2m\theta+\sin(m-n)\theta=0.$

(18) $\left.\begin{aligned}\sin\{\pi x(x+y)\}+\sin\{\pi y(x+y)\}&=0,\\ \sin\pi x^2+\sin\pi y^2&=0.\end{aligned}\right\}$

(19) Trace the changes in the sign and magnitude of the following expressions, as $\theta$ changes from 0 to $\pi$.

| | | | |
|---|---|---|---|
| (i) | $2\sin\theta\,.\,\cos\theta.$ | (ii) | $\cos^2\theta-\sin^2\theta.$ |
| (iii) | $\sin 3\theta.$ | (iv) | $\cot 2A.$ |
| (v) | $\sin(\theta+\alpha).$ | (vi) | $\cos(2\theta-\alpha).$ |

(20) Explain why the equations

$$\theta=n\pi+(-1)^n\alpha \quad\text{and}\quad \frac{\pi}{2}-\theta=2n\pi\pm\left(\frac{\pi}{2}-\alpha\right)$$

have exactly the same series of solutions.

(21) Explain why exactly the same series of angles are given by the two equations $\theta+\dfrac{\pi}{4}=n\pi+(-1)^n\dfrac{\pi}{6}$, and $\theta-\dfrac{\pi}{4}=2n\pi\pm\dfrac{\pi}{3}$.

## CHAPTER XIV.

### ON LOGARITHMS.

189. IN Algebra it is explained

(i) that the *multiplication* of different powers of the same quantity is effected by *adding* the *indices* of those powers;

(ii) that *division* is effected by *subtracting* the *indices;*

(iii) that *involution* and *evolution* are respectively effected by the *multiplication* and *division* of the *indices.*

*Example* 1. If $m=a^h$, $n=a^k$,

then

$$m \times n = a^h \times a^k = a^{h+k} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

$$m \div n = a^h \div a^k = a^{h-k} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

$$\left.\begin{array}{l} m^3 = (a^h)^3 = a^{3h}, \\ \sqrt[4]{m} = m^{\frac{1}{4}} = (a^h)^{\frac{1}{4}} = a^{\frac{h}{4}} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

*Example* 2. If $347 = 10^{2\cdot5403295}$* and $461 = 10^{2\cdot6637009}$, prove that $347 \times 461 = 10^{5\cdot2040304}$.

We have

$$347 \times 461 = 10^{2\cdot5403295} \times 10^{2\cdot6637009}$$

$$= 10^{2\cdot5403295+2\cdot6637009}$$

$$= 10^{5\cdot2040304}. \qquad \text{Q. E. D.}$$

---

* The number 347 lies between 100 and 1000, i.e. between $10^2$ and $10^3$. Hence, if there is a power of 10 which is equal to 347, its index must be greater than 2 and less than 3, i.e. equal to 2+a fraction.

## EXAMPLES. LI.

(1) If $m=a^h$, $n=a^k$, express in terms of $a$, $h$ and $k$,

(i) $m^2\times n^3$. (ii) $m^4\div n^5$. (iii) $\sqrt[3]{m^4\times n^5}$. (iv) $\{\sqrt[4]{m^5\times n^3}\}^2$.

(2) If $453=10^{2\cdot6560982}$ and $650=10^{2\cdot8129134}$, find the indices of the powers of 10 which are equal to

(i) $453\times650$. (ii) $(453)^4$. (iii) $650^3\times453^2$. (iv) $\sqrt[3]{453}$.

(v) $\sqrt{453}\times\sqrt[6]{650}$. (vi) $\sqrt[5]{453}\times(650)^3$. (vii) $\sqrt{453\times650}$.

(3) Express in powers of 2 the numbers, 8, 32, $\frac{1}{2}$, $\frac{1}{16}$, ·125, 128.

(4) Express in powers of 3 the numbers, 9, 81, $\frac{1}{3}$, $\frac{1}{27}$, ·i, $\frac{1}{81}$.

190. Suppose that some convenient number (such as 10) having been chosen, we are given a list of the indices of the powers of that number, which are equivalent to every whole number from 1 up to 100000

Such a list could be used to shorten Arithmetical calculations.

*Example* 1. Multiply 3759 by 4781 and divide the result by 2690

Looking in our *list* we should find $3759=10^{3\cdot5750723}$, $4781=10^{3\cdot6795187}$, $2690=10^{3\cdot4297523}$.

Therefore $3759\times4781\div2690=10^{3\cdot5750723}\times10^{3\cdot6795187}\div10^{3\cdot4297523}$ $=10^{3\cdot5750723+3\cdot6795187-3\cdot4297523}=10^{3\cdot8248387}$.

The list will give us that $10^{3\cdot8248387}=6680\cdot9$.

Therefore the answer correct to five significant figures is 6680·9.

*Example* 2. Simplify $3^6\times2^{10}\div\sqrt[3]{17601}$.

The *list* gives $2=10^{\cdot3010300}$, $3=10^{\cdot4771213}$ and $17601=10^{4\cdot2455373}$.

Thus $3^6 \times 2^{10} \div \sqrt[3]{17601} = (10^{\cdot 4771213})^6 \times (10^{\cdot 3010300})^{10} \div (10^{4 \cdot 2455373})^{\frac{1}{3}}$

$= 10^{2 \cdot 8627278} \times 10^{3 \cdot 0103000} \div 10^{1 \cdot 4151791}$

$= 10^{2 \cdot 8627278 + 3 \cdot 0103000 - 1 \cdot 4151791}$

$= 10^{4 \cdot 4578487}$.

And from our list we find $10^{4 \cdot 4578487} = 28697$, nearly.

## EXAMPLES. LII.

Given that $2 = 10^{\cdot 3010300}$, $3 = 10^{\cdot 4771213}$ and $7 = 10^{\cdot 8450980}$, find the indices of the powers of 10 equivalent to the quantities in the first 6 examples.

(1) $2^2, 3^2, 2^3, 2 \times 3, 2^4, 7^2$.

(2) 14, 16, 18, 24, 27, 42.

(3) 10, 5, 15, 25, 30, 35.

(4) 36, 40, 48, 50, 200, 1000.

(5) $3^{10} \times 7^{10} \div 2^{20}$, $2^{12} \times 3^{20} \div 7^{11}$.

(6) $\sqrt[3]{21} \times \sqrt[4]{18}$, $\sqrt[2]{\cdot 49 \times 4^5} \times \sqrt[3]{3^4 \times 2^{10}}$.

(7) Find approximately the numerical value of $\sqrt[10]{42}$ having given that $10^{\cdot 1623249} = 1 \cdot 4532$ nearly.

(8) Find approximately the numerical value of $\sqrt[3]{(42)^4} \times \sqrt[4]{(42)^3}$ having given that $10^{3 \cdot 38177} = 2408 \cdot 6$.

(9) Find the value (i) of $\sqrt[3]{6} \times \sqrt[4]{7} \times \sqrt[5]{9}$, (ii) of $\sqrt[10]{2} \times 3^{-\frac{5}{4}} \times 7^{\frac{7}{11}}$ having given that $10^{\cdot 6615067} = 4 \cdot 5868$ and $10^{-\cdot 0285094} = \cdot 93646$.

(10) Find the value of $(67 \cdot 21)^{\frac{3}{5}} \times (49 \cdot 62)^{\frac{1}{5}} \times (3 \cdot 971)^{-\frac{7}{5}}$ having given that $67 \cdot 21 = 10^{1 \cdot 8274339}$, $49 \cdot 62 = 10^{1 \cdot 6956568}$, $3 \cdot 971 = 10^{\cdot 5988999}$ and $10^{\cdot 5971310} = 3 \cdot 9549$.

(11) Find the area of a square field whose side is 640·12 feet; having given that $640 \cdot 12 = 10^{2 \cdot 8062614}$ and that $10^{5 \cdot 6125228} = 40975 \cdot 3$.

(12) Find the edge of a solid cube which contains 42601 cubic inches; having given $42601 = 10^{4 \cdot 6294198}$ and $10^{1 \cdot 5431399} = 34 \cdot 925$.

(13) Find the edge of a solid cube which contains 34·701 cubic inches; having given that $34 \cdot 701 = 10^{1 \cdot 5403420}$, and $10^{\cdot 5134473} = 3 \cdot 2617$.

(14) Find the volume of the cube the length of one of whose edges is 47·931 yards; having given that $47 \cdot 931 = 10^{1 \cdot 6806165}$ and that $10^{5 \cdot 0418495} = 110115$.

191. The powers of any other number than 10 might be used in the manner explained above, but 10 is the most *convenient* number, as will presently appear.

192. This method, in which the *indices* of the powers of a certain fixed number (such as 10) are made use of, is called the *Method of Logarithms.*

**Indices** thus used are called **logarithms.**

The *fixed number* whose powers are used is called the **base.** Hence we have the following definition:

**DEF.** The **logarithm** of a number to a given *base*, is the **index** of that power of the base, which is equal to the given number.

Thus, if $l$ be the logarithm of the number $n$ to the base $a$, then

$$a^l = n.$$

193. The **notation** used is $\log_a n = l.$

Here, $\log_a n$ is an abbreviation for the words 'the logarithm of the number $n$ to the base $a$.' And this means, as we have explained above, 'the **index** of that power of $a$ which is equal to the number $n$.'

*Example* 1. What is the logarithm of $a^{\frac{3}{2}}$ to the base $a$?

That is, what is the index of the power of $a$ which is $a^{\frac{3}{2}}$?

The index is $\frac{3}{2}$; therefore $\frac{3}{2}$ is the required logarithm, or

$$\log_a a^{\frac{3}{2}} = \tfrac{3}{2}.$$

*Example* 2. What is the logarithm of 32 to the base 2?

That is, what is the **index** of the power of 2 which is equal to 32?

Now $32 = 2^5$. $\therefore$ the required index is 5; or $\log_2 32 = 5.$

The use of Logarithms is based upon the following propositions:—

**I.** The logarithm of the product of two numbers is equal to the logarithm of one of the numbers + the logarithm of the other.

For, let $\log_a m = x$ and $\log_a n = y$,

then, $\log_a (m \times n) = \log_a (a^x \times a^y) = \log_a (a^{x+y}) = x + y$
$= \log_a m + \log_a n.$

**II.** The logarithm of the quotient of two numbers is the logarithm of the dividend − the logarithm of the divisor.

For, $\log_a \left(\frac{m}{n}\right) = \log_a \left(\frac{a^x}{a^y}\right) = \log_a (a^{x-y}) = x - y$ [as above]
$= \log_a m - \log_a n.$

**III.** The logarithm of a number raised to a power $k$ is $k$ times the logarithm of the number.

For, $\log_a (m^k) = \log_a \{(a^x)^k\} = \log_a (a^{kx}) = kx$
$= k$ times $\log_a m.$

*Examples. Given*

$\log_{10} 2 = {\cdot}3010300$, $\log_{10} 3 = {\cdot}4771213$, $\log_{10} 7 = {\cdot}8450980$

*find the values of the following:*

(i) $\log_{10} 6 = \log_{10} (2 \times 3) = \log_{10} 2 + \log_{10} 3$ [by I.]
$= {\cdot}3010300 + {\cdot}4771213$
$= {\cdot}7781513.$

(ii) $\log_{10} \frac{7}{3} = \log_{10} 7 - \log_{10} 3 = {\cdot}8450980 - {\cdot}4771213$ [by II.]
$= {\cdot}3679767.$

(iii) $\log_{10} 3^5 = 5$ times $\log_{10} 3 = 5 \times {\cdot}4771213$ [by III.]
$= 2{\cdot}3856065.$

(iv) $\log_{10} \sqrt[3]{\frac{3 \times 4}{7}} = \log_{10} \left(\frac{3 \times 4}{7}\right)^{\frac{1}{3}} = \frac{1}{3}$ of $\log_{10} \frac{3 \times 4}{7}$ [by III.]
$= \frac{1}{3}$ of $(\log 3 + \log 4 - \log 7)$ [by I. and II.]
$= \frac{1}{3}$ of $\{{\cdot}4771213 + \text{twice } {\cdot}30103 - {\cdot}8450980\}$
$= \frac{1}{3}$ of ${\cdot}2340833 = {\cdot}0780278.$

(v) $\log_{10} 5 = \log_{10} \frac{10}{2} = \log_{10} 10 - \log_{10} 2 = 1 - {\cdot}3010300$
$= {\cdot}6989700.$

## EXAMPLES. LIII.

(1) Find the logarithms to the base $a$ of $a^3$, $a^{\frac{10}{3}}$, $\sqrt[4]{a}$, $\sqrt[3]{a^2}$, $\frac{1}{a^{\frac{5}{2}}}$.

(2) Find the logarithms to the base 2 of 8, 64, $\frac{1}{2}$, $\cdot 125$, $\cdot 015625$, $\sqrt[3]{64}$.

(3) Find the logarithms to the base 3 of 9, 81, $\frac{1}{3}$, $\frac{1}{27}$, $\cdot\dot{1}$, $\frac{1}{81}$.

(4) Find the logarithms to base 4 of 8, $\sqrt[3]{16}$, $\sqrt[4]{\cdot 5}$, $\sqrt[3]{\cdot 015625}$.

(5) Find the value of

$$\log_2 8,\ \log_2 \cdot 5,\ \log_3 243,\ \log_5 (\cdot 04),\ \log_{10} 1000,\ \log_{10} \cdot 001.$$

(6) Find the value of

$$\log_a a^{\frac{1}{3}},\ \log_b \sqrt[3]{b^2},\ \log_8 2,\ \log_{27} 3,\ \log_{100} 10.$$

Given that

$\log_{10} 2 = \cdot 3010300$, $\log_{10} 3 = \cdot 4771213$ and $\log_{10} 7 = \cdot 8450980$,

find the values of

(7) $\log_{10} 6$, $\log_{10} 42$, $\log_{10} 16$. (8) $\log_{10} 49$, $\log_{10} 36$, $\log_{10} 63$.

(9) $\log_{10} 200$, $\log_{10} 600$, $\log_{10} 70$. (10) $\log_{10} 5$, $\log_{10} 3\cdot\dot{3}$, $\log_{10} 50$.

(11) $\log_{10} 35$, $\log_{10} 150$, $\log_{10} \cdot 2$. (12) $\log_{10} 3\cdot 5$, $\log_{10} 7\cdot 29$, $\log_{10} \cdot 081$.

(13) Given $\log_{10} 2$, $\log_{10} 3$, $\log_{10} 7$, find the value (i) of $\sqrt[3]{6} \times \sqrt[4]{7} \times \sqrt[5]{9}$, (ii) of $\sqrt[10]{2} \times 3^{-\frac{5}{4}} \times 7^{\frac{7}{11}}$

$[\cdot 6615067 = \log_{10} 4\cdot 5868;\ -\cdot 0285094 = \log_{10} \cdot 93646]$.

(14) Prove that (i) $\log\{\sqrt[3]{2} \times \sqrt[4]{7} \div \sqrt[5]{9}\} = \frac{1}{3}\log 2 + \frac{1}{4}\log 7 - \frac{2}{5}\log 3$,

(ii) $\log\{\sqrt[10]{2} \times 3^{-\frac{5}{4}} \times 7^{\frac{7}{11}}\} = \frac{1}{10}\log 2 - \frac{5}{4}\log 3 + \frac{7}{11}\log 7$.

(15) If $\log_{10} a = 2\cdot 6560982$ and $\log_{10} b = 2\cdot 8129134$, show that

(i) $\log_{10} ab = 5\cdot 4690116$, (ii) $\log_{10} a^4 = 10\cdot 6243928$,
(iii) $\log_{10} a^2b^3 = 13\cdot 7509366$, (iv) $\log_{10} \sqrt[3]{a} = \cdot 8853661$.
(v) $\log_{10} (a^3b)^{\frac{1}{6}} = 1\cdot 7968680$, (vi) $\log_{10} a^{\frac{1}{5}} b^3 = 8\cdot 9699598$.

(16) Show (i) that $\log_{10} \sqrt[3]{21} \times \sqrt[4]{18} = \cdot 7545579$, (ii) that

$$\log_{10} \sqrt[2]{(\cdot 49 \times 4^5)} \times \sqrt[3]{(3^4 \times 2^{10})} = 2\cdot 989843.$$

## COMMON LOGARITHMS.

194. That System of Logarithms whose base is 10, is called the Common System of Logarithms.

In speaking of logarithms hereafter, *common* logarithms are referred to unless the contrary is expressly stated.

195. We shall assume that a power of ten can be found which is practically equivalent to any number.

196. The indices of these powers of 10, *i.e.* the Common Logarithms, are in general *incommensurable* numbers.

Their value for every whole number, from 1 to 100000, has been calculated to 7 significant figures. Thus any calculation made with the aid of logarithms is as exact as the most carefully observed measurement (cf. Arts. 17, 216).

197. Now, the greater the index of any power of 10, the greater will be the numerical value of that power; and the less the index, the less will be the numerical value of the power.

Hence, if one number be less than another, the logarithm of the first will be less than the logarithm of the second.

But the student should notice that logarithms (or indices) are *not proportional* to the corresponding numbers.

*Example.* 1000 is less than 10000; and the logarithm to base 10 of the first is 3 and of the second is 4

But 1000, 10000, 3, 4 are not in proportion.

198. We know from Algebra that

$$1 = 10^0,$$

$$\begin{array}{rcr}
10 = 10^1 & \text{and that} & \cdot 1 = \frac{1}{10} = 10^{-1} \\
100 = 10^2 & \ldots\ldots\ldots & \cdot 01 = \frac{1}{100} = 10^{-2} \\
1000 = 10^3 & \ldots\ldots\ldots & \cdot 001 = \frac{1}{1000} = 10^{-3} \\
10000 = 10^4 & \ldots\ldots\ldots & \cdot 0001 = \frac{1}{10000} = 10^{-4}
\end{array}$$

and so on.

Hence, the logarithm of 1 is 0.

The (common) logarithm of any number greater than 1 is *positive.*

The logarithm of any positive number less than 1 is *negative.*

199. We observe also

that the logarithm of any number between 1 and 10 is a positive decimal fraction;

that the logarithm of any number between 10 and 100, *i.e.* between $10^1$ and $10^2$, is of the form 1 + a decimal fraction;

that the logarithm of any number between 1000 and 10000, *i.e.* between $10^3$ and $10^4$, is of the form 3 + a decimal fraction;

and so on.

200. We observe also

that the logarithm of any number between 1 and ·1, *i.e.* between $10^0$ and $10^{-1}$, can be written in the form – 1 + a decimal fraction;

that the logarithm of any number between ·1 and ·01, *i.e.* between $10^{-1}$ and $10^{-2}$, can be written in the form – 2 + a decimal fraction;

and so on.

*Example* 1. How many digits are there in the integral part of the number whose logarithm is 3·67192?

We know that $3 = \log 1000$ and $4 = \log 10000$;

$\therefore$ $3{\cdot}67192 = \log$ of some number between 1000 and 10000; that is, the integral part of the number has 4 digits.

*Example* 2. Given that $\log 3 = {\cdot}4771213$, find the number of the digits in the integral part of $3^{20}$.

Here, $\log(3^{20}) = 20$ times $\log 3 = 9{\cdot}542426$;

hence, $\log(3^{20})$ lies between 9 and 10;

therefore, as in Example 1, the number lies between $10^9$ and $10^{10}$; that is, its integral part has 10 digits.

*Example* 3. Supposing that the decimal part of the logarithm is to be kept positive, find the integral part of the logarithm of ·0001234.

This number is greater than ·0001 *i.e.* than $10^{-4}$ and less than ·001, *i.e.* than $10^{-3}$.

Therefore its logarithm lies between $-3$ and $-4$, and therefore it is $-4 +$ a fraction; the integral part is therefore $-4$.

## EXAMPLES. LIV.

Note. The *decimal* part of a logarithm is to be kept *positive*.

(1) Write down the integral part of the common logarithms of 17601, 361·1, 4·01, 723000, 29.

(2) Write down the integral part of the common logarithms of ·04, ·0000612, ·7963, ·001201. (See Note above.)

(3) Write down the integral part of the common logarithms of 7963, ·1, 2·61, 79·6341, 1·0006, ·00000079.

(4) How many digits are there in the integral part of the numbers whose common logarithms are respectively

3·461, ·3020300, 5·4712301, 2·6710100?

(5) Give the position of the first *significant* figure in the numbers whose logarithms are

$$-2+\cdot4612310,\quad -1+\cdot2793400,\quad -6+\cdot1763241.$$

(6) Give the position of the first *significant* figure in the numbers whose common logarithms are $4\cdot2990713$, $\cdot3040595$, $2\cdot5860244$, $-3+\cdot1760913$, $-1+\cdot3180633$, $\cdot4980347$.

(7) Given $\log_{10}2=\cdot30103$, find the number of digits in the integral part of $8^{10}$, $2^{12}$, $16^{20}$, $2^{100}$.

(8) Given that $\log 7=\cdot8450980$, find the number of digits in the integral part of $7^{10}$, $49^{6}$, $343^{\frac{100}{3}}$, $(\frac{10}{7})^{20}$, $(4\cdot9)^{12}$, $(3\cdot43)^{10}$.

(9) Find the position of the first significant figure in

$$\sqrt[10]{2},\ (\tfrac{1}{2})^{10},\ (\tfrac{10}{7})^{20},\ (\cdot02)^{4},\ (\cdot49)^{6}.$$

(10) Find the position of the first significant figure in the numerical value of

$$20^{7},\ (\cdot02)^{7},\ (\cdot007)^{2},\ (3\cdot43)^{\frac{1}{10}},\ (\cdot0343)^{8},\ (\cdot0343)^{\frac{1}{10}}.$$

201. Prop. *To prove that if two numbers expressed in the decimal notation have the same digits (so that they differ only in the position of the decimal point), their logarithms to the base* 10 *will differ only by an integer.*

The decimal point in a number is moved by multiplying or dividing the number by some *integral* power of 10.

Let the numbers be $m$ and $n$; then $m=n\times10^{k}$, where $k$ is a whole number positive or negative; and

$$\log m=\log(n\times10^{k})=\log n+\log10^{k}$$
$$=\log n+k.$$

*Example* 1. Let the numbers be $1\cdot2345$ and $1234\cdot5$.

Then, $\qquad 1234\cdot5=1\cdot2345\times10^{3}$,

therefore $\qquad \log(1234\cdot5)=\log(1\cdot2345)+3.$

*Example* 2. Given $\log 1{\cdot}7692 = {\cdot}24776$, find

(i) $\log 17692$, (ii) $\log {\cdot}0017692$, (iii) $176{\cdot}92$.

Here, $$\log 17692 = \log \{(1{\cdot}7692) \times 10^4\} = \log (1{\cdot}7692) + 4$$
$$= {\cdot}24776 + 4 = 4{\cdot}24776.$$
$$\log {\cdot}0017692 = \log \{(1{\cdot}7692) \times 10^{-3}\} = -3 + {\cdot}24776.$$
$$\log 176{\cdot}92 = \log \{(1{\cdot}7692) \times 10^2\} = 2{\cdot}24776.$$

**202.** It is *convenient* to keep the *decimal part* of common logarithms always *positive*, because then the *decimal part* of the logarithms of any numbers expressed by the same digits will be *always* the same.

**203.** The decimal part of a logarithm is called the **mantissa.**

**204.** The integral part is called the **characteristic.**

**205.** The characteristic of a logarithm can be always obtained by the following rule, which is evident from page 167.

**RULE.** The characteristic of the logarithm of a number greater than unity is **one less** than the number of *integral* figures in that number.

The characteristic of a number less than unity is negative, and (when the number is expressed as a decimal,) is **one more** than the number of cyphers between the decimal point and the first significant figure to the right of the decimal point.

**206.** When the characteristic is negative, as for example in the logarithm $-3 + {\cdot}1760913$, the logarithm is abbreviated thus, $\bar{3}{\cdot}1760913$.

*Example* 1. The characteristics of 36741, 36·741, ·0036741, 3·6741 and ·36741 are respectively 4, 1, $-3$, 0, and $-1$.

*Example* 2. Given that the mantissa of the logarithm of 36741 is 5651510, we can at once write down the logarithm of any number whose digits are 36741.

Thus
$$\begin{aligned}\log 3674100 &= 6{\cdot}5651510,\\ \log 36741 &= 4{\cdot}5651510,\\ \log 367{\cdot}41 &= 2{\cdot}5651510,\\ \log {\cdot}36741 &= \bar{1}{\cdot}5651510,\\ \log {\cdot}00036741 &= \bar{4}{\cdot}5651510,\end{aligned}$$
and so on.

207. In any set of tables of common logarithms the student will find the *mantissa only* corresponding to any set of digits.

It would obviously be superfluous to give the *characteristic.*

208. It is most important to remember to keep the **mantissa** always **positive.**

*Example.* Find the fifth root of ·00065061.

Here $\log_{10} {\cdot}00065061 = \bar{4}{\cdot}8133207,$

$$\therefore \log_{10} ({\cdot}00065061)^{\frac{1}{5}} = \tfrac{1}{5}(\bar{4}{\cdot}8133207) = \tfrac{1}{5}(-4 + {\cdot}8133207)$$
$$= \tfrac{1}{5}(-5 + 1{\cdot}8133207) = -1 + {\cdot}3626641 = \bar{1}{\cdot}3626641,$$

and $\bar{1}{\cdot}3626641 = \log {\cdot}23050,$

$\therefore$ the fifth root of ·00065061 = ·23050 nearly.

## EXAMPLES. LV.

(1) Write down the logarithms of 776·43, 7·7643, ·00077643 and 776430. (The table gives opposite the numbers 77643, the figures 8901023.)

(2) Given that $\log_{10} 59082 = 4{\cdot}7714552$, write down the logarithms of 5908200, 5·9082, ·00059082, 590·82 and 5908·2.

(3) Find the fourth root of ·0059082, having given that

$$\log 5{\cdot}9082 = {\cdot}7714552;\quad 4{\cdot}4428638 = \log_{10} 27724.$$

(4) Find the product of ·00059082 and ·027724, having given that ·21431 = log 16380 (cf. Question 3).

(5) Find the 10th root of ·077643 (cf. Question 1), having given that ·8890102 = log 7·7448.

(6) Find the product of $({\cdot}27724)^2$ and ·077643. (See Questions 1 and 3; 7758288 = log 59680.)

* * 209. *To transform a system of logarithms having a given base, to another system with a different base.*

If we are given a list of logarithms calculated to a given base, we can deduce from it a list of logarithms calculated to any other base.

Let $a$ be the given base; let $b$ be any other base.

Let $m$ be any number. Then the logarithm of $m$ to the base $a$ is in the given list. Let this logarithm be $l$. Then

$$m = a^l.$$

We wish to find the logarithm of $m$ to the base $b$. Let it be $x$.

Then $$m = b^x. \quad \text{But } m = a^l;$$

$$\therefore\ a^l = b^x;\ \text{or, } b = a^{\frac{l}{x}};$$

or, $$\frac{l}{x} \text{ is the logarithm of } b \text{ to the base } a.$$

Now the logarithm of $b$ to the base $a$ is given. For it is in the list of logarithms to the base $a$.

Thus, $$\frac{l}{x} = \log_a b;\ \text{or, } x = \frac{l}{\log_a b};$$

or, $$\log_b m = \frac{\log_a m}{\log_a b}.$$

Hence, to calculate the logarithms of a series of numbers to a new base $b$, we have only to divide each of the logarithms of the numbers to any given base $a$, by a certain divisor, viz. $\log_a b$.

If then a list of logarithms to some base $e$ can be made, we can deduce from it a list of common logarithms, by multiplying each logarithm in the given list by $\frac{1}{\log_e 10}$.

*Example.* Show how to transform logarithms, having 5 for base to logarithms having 125 for base.

Suppose $m=5^l$, so that $l=\log_5 m$.

Now $125=5^3$, so that $3=\log_5 125$,

and $m=5^{3\times\frac{l}{3}}=125^{\frac{l}{3}}$, so that $\frac{l}{3}=\log_{125} m$.

Thus the logarithm of any number to base 5, divided by 3 (*i.e.* by $\log_5 125$), is the logarithm of the same number to the base 125.

**210. The student will find that the logarithms of numbers cannot be calculated to the base 10 directly.

They are first calculated to the base 2·7182818, etc., which is the sum of the series

$$1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\text{etc. } ad\ inf.$$

This number is called $e$.

And the constant divisor in this case is $\log_e 10$,

and $$\frac{1}{\log_e 10}=\frac{1}{2{\cdot}30258509}={\cdot}43429448, \text{ etc.}$$

When this constant divisor is transformed into a *multiplier*, this constant **multiplier** is called a **modulus.**

*Example.* Given that $\log_{10} 12=1{\cdot}0791812$, shew how to transform common logarithms to logarithms having 12 for base.

Here $$10^{1{\cdot}0791812}=12;$$

$$\therefore\ 10=12^{\frac{1}{1{\cdot}0791812}}=12^{{\cdot}92662,\text{ etc.}}$$

$$\therefore\ 10^h=12^{h\times{\cdot}92662}.$$

## EXAMPLES. LVI.

Show how to transform

(1) Logarithms with base 2 to logarithms with base 8.

(2) Logarithms with base 9 to logarithms with base 3.

(3) Common logarithms to logarithms with base 2.

(4) Logarithms with base 3 to common logs.

(5) Common logs to logarithms with base 3.

(6) Given $\log_{10} 2 = \cdot 3010300$, find $\log_2 10$.

(7) Given $\log_{10} 7 = \cdot 8450980$, find $\log_7 10$.

(8) Given $\log_{10} 2 = \cdot 3010300$, find $\log_8 10$ and $\log_{32} 10$.

## * * MISCELLANEOUS EXAMPLES. LVII.

(1) Find $\log_2 8$, $\log_5 1$, $\log_8 2$, $\log_7 1$, $\log_{32} 128$

(2) Show that the logarithms of all except eight of the numbers from 1 to 30 inclusive, can be calculated in terms of log 2, log 3 and log 7.

(3) Show that the logarithms of the numbers 1 to 10 inclusive may be found in terms of the logarithms of 8, 14, 21.

(4) The mantissa of the log of 85762 is 9332949. Find the log of $\sqrt[11]{\cdot 0085762}$.

Find how many figures there are in $(85762)^{11}$, when it is multiplied out.

(5) Find the product of $47\cdot 609$, $476\cdot 09$, $\cdot 47609$, $\cdot 000047609$, having given that $\log 4\cdot 7609 = \cdot 6776891$ and $\cdot 7107564 = \log 5\cdot 1375$.

(6) What are the characteristics of the logarithm of 3742 to the bases 3, 6, 10 and 12 respectively.

(7) Having given that $\log 2 = \cdot 3010300$, $\log 3 = \cdot 4771213$ and $\log 7 = \cdot 8450980$, solve the following equations :

(i) $2^x \times 3^{4x} = 7^2$, (ii) $3^{2x} = 128 \times 7^{4-x}$,

(iii) $12^x = 49$, (iv) $2^{8x} = 21^{4-3x}$.

(8) Given $\log_{10} 7$, find $\log_7 490$.

(9) Given $\log_{10} 3$, find $\log_9 270$.

(10) Given $\log_{10} 2$, find $\log_5 10$.

(11) Given $\log_8 9 = a$, $\log_2 5 = b$, $\log_5 7 = c$; find the logs to base 10 of numbers 1 to 7 inclusive.

(12) How many positive integers are there whose logarithms to base 2 have 5 for a characteristic?

(13) If $a$ be an integer, how many positive integers are there whose logs to base $a$ have 10 for their characteristic?

(14) Given log 2 and log 7, find the eleventh root of $(39{\cdot}2)^2$.

$$\log 1{\cdot}9485 = {\cdot}289688.$$

(15) Prove that $7 \log \frac{15}{16} + 6 \log \frac{8}{3} + 5 \log \frac{2}{5} + \log \frac{32}{25} = \log 3$.

(16) Prove that

$$2 \log a + 2 \log a^2 + 2 \log a^3 \ldots + 2 \log a^n = n\,(n+1) \log a.$$

(17) Prove that $\log_a b \,.\, \log_b a = 1$; and that $\log_a b \,.\, \log_b c \,.\, \log_c a = 1$.

(18) Prove that $\log_a r = \log_a b \,.\, \log_b c \,.\, \log_c d \ldots \log_q r$.

(19) Given that the integral part of $(3{\cdot}456)^{100000}$ contains 53856 digits, find log $345{\cdot}6$ correct to five places of decimals.

(20) Given that the integral part of $(3{\cdot}981)^{100000}$ contains sixty thousand digits, find log 39810 correct to five places of decimals.

(21) If the number of births in a year be $\frac{1}{48}$ of the population at the beginning of the year, and the number of deaths $\frac{1}{60}$, find in what time the population will be doubled.

Given log 2, log 3, and that $\log 241 = 2{\cdot}3820170$.

(22) Prove that

$$\log s + \log (s - a) - \log b - \log c = 2 \log \sqrt{\frac{s\,(s-a)}{bc}}.$$

(23) Prove that

$$\log (a^2 + x^2) + \log (a + x) + \log (a - x) = \log (a^4 - x^4).$$

(24) Prove that

$$\log \sin 4A = \log 4 + \log \sin A + \log \cos A + \log \cos 2A.$$

# CHAPTER XV.

## ON THE USE OF MATHEMATICAL TABLES.

213. The Logarithms referred to in this chapter, and in future throughout the book, are *Common* Logarithms.

214. Books of Mathematical Tables usually give an explanation of their own contents, but there are some points common to all such Tables which we proceed to explain.

215. The student will be supposed to have access to a book containing the following:

(i) A list of the logarithms of all whole numbers from 1 to 99999, calculated to seven significant figures;

(ii) A list of the numerical values, calculated to seven significant figures, of the Trigonometrical Ratios of all angles, between 0° and 90°, which differ by 1′;

(iii) A list of the logarithms of these Ratios calculated to seven significant figures.

These will be found in Chambers' *Mathematical Tables*.

At the end of this Chapter we give the logarithms to five figures of all numbers from 100 to 999. This Table will be found useful in questions involving certain kinds of numerical calculations.

216. We have said that logarithms are in general incommensurable numbers. Their values can therefore only be given approximately.

If the value of any number is given to seven significant figures, then the **error** (i.e. the difference between the *given* value and the *exact* value of the number) is less than a millionth part of the number.

*Example.* 3·141592 is the value of $\pi$ correct to seven significant figures. The *error* is less than ·000001; for $\pi$ is less than 3·141593, and greater than 3·141592.

The ratio of ·000001 to 3·141592 is equal to 1 : 3141592. The ratio of ·000001 to $\pi$ is less than this; i.e. much less than the ratio of one to one million.

217. An actual measurement of any kind must be made with the greatest care, with the most accurate instruments, by the most skilful observers, if it is to attain to anything like the accuracy represented by 'seven significant figures.'

Therefore the value of any quantity given correct to 'seven significant figures' is exact for all practical purposes.

218. We are given in the Tables the logarithms of all numbers from 1 to 99999; that is, of any number having *five* significant figures.

A Table consisting of the logarithms of all numbers from 1 to 9999999 (i.e. of any number having *seven* significant figures) would be *a hundred times as large.*

219. There is however a rule by which, if we are given a *complete* list of the logarithms of numbers having *five* significant figures, we can find the logarithms of numbers having *six* or *seven* significant figures.

*Example.* Suppose we require the logarithm of 4·804213.

From the Tables we find

$$\log 4{\cdot}8042 = {\cdot}6816211, \qquad \text{i.e. } 4{\cdot}8042 = 10^{{\cdot}6816211\ldots},$$
$$\log 4{\cdot}8043 = {\cdot}6816301, \qquad 4{\cdot}8043 = 10^{{\cdot}6816301\ldots}.$$

The number 4·804213 lies *between* the two numbers 4·8042, 4·8043 whose logarithms are found in the Tables, so that the required logarithm must lie *between* the two given logarithms.

Therefore we suppose that

$$\log 4{\cdot}804213 = {\cdot}6816211 + d, \qquad \text{i.e. } 4{\cdot}804213 = 10^{{\cdot}6816211\ldots+d}.$$

220. The **RULE** is as follows. **The differences between three numbers are proportional to the corresponding differences between the logarithms of those numbers,** provided that the *differences* between the numbers are *small* compared with the numbers.

*Example.* *Given* $\log 48042 = 4{\cdot}6816211$
$\log 48043 = 4{\cdot}6816301,$
*find* $\log 48042{\cdot}13$.

We may state the question thus.

*Given* $\log 48042 = 4{\cdot}6816211$
$\log (48042 + 1) = 4{\cdot}6816211 + {\cdot}0000890,$
*find* $\log (48042 + {\cdot}13)$.

Let $\log (48042 + {\cdot}13 = 4{\cdot}6816211 + d.$

Thus corresponding to the **differences** 1 and ·13 in the numbers we have the **differences** ·0000090 and $d$ in the logarithms.

But these differences are in proportion,

$$\therefore\ d = \frac{{\cdot}13}{1} \text{ times } {\cdot}000009$$
$$= \tfrac{13}{100} \text{ of } {\cdot}000009 = {\cdot}00000117\ldots;$$
$$\therefore\ \log 48042{\cdot}13 = 4{\cdot}6816211 + {\cdot}00000117$$
$$= 4{\cdot}68162227 = 4{\cdot}6816223 \text{ (to seven figures).}$$

221. We shall refer to the above rule as the **Rule of Proportional Differences.**

It is often called also 'The Principle of Proportional Parts.'

222. In Art. 197 we said that numbers are *not* proportional to their Logarithms. Hence the differences of numbers and the corresponding differences of their logarithms cannot be *exactly* in proportion. The rule is however true for all practical purposes. The proof of the rule belongs to a higher part of the subject than the present.

223. In the above example we said that

$$6{\cdot}68162227 = 6{\cdot}6816223;$$

and for this reason. We are *retaining* only *seven* significant figures in the decimal part of the logarithm.

If we put 6·6816222 for 6·68162227 the 'error' is greater than ·00000007.

If we put 6·6816223 for 6·68162227 the 'error' is less than ·00000003.

Thus the second error is less than the first.

In such a case, 1 must be added to the last digit which is retained, when the first digit which is neglected is 5 or greater than 5.

224. We give two more specimen examples.

*Example* 1. *Find the logarithm of* ·004804213.

We first find as before, by the rule of proportional differences, that

$$\log 4{\cdot}804213 = {\cdot}6816223$$

$$\therefore \log {\cdot}004804213 = \bar{3}{\cdot}6816223.$$

*Example* 2. *Find the number whose logarithm is* 2·5354291.

In the Table we find that

$$\cdot 5354207 = \log 3\cdot 4310 \quad \text{..............................(i)},$$

and

$$\cdot 5354334 = \log 3\cdot 4311 \quad \text{..............................(ii)}.$$

Let

$$\cdot 5354291 = \log (3\cdot 4310 + d) \quad \text{.....................(iii)}.$$

Here we have three logarithms and three numbers.

Subtracting

$$\begin{array}{r} \cdot 5354334 \\ \cdot 5354207 \\ \hline \cdot 0000127 \end{array} \qquad \begin{array}{r} \cdot 5354291 \\ \cdot 5354207 \\ \hline \cdot 0000084 \end{array}$$

The corresponding differences are ·0001 and $d$.

Hence,

$$d = \frac{\cdot 0000084}{\cdot 0000127} \text{ of } \cdot 0001$$

$$= \tfrac{84}{127} \text{ of } \cdot 0001$$

$$= \cdot 0000661.$$

Therefore from (iii) $\cdot 5354291 = \log (3\cdot 4310 + \cdot 000066\ldots)$

$$= \log 3\cdot 431066.$$

Hence,

$$2\cdot 5354291 = \log 343\cdot 1066,$$

or, the required number is 343·1066.

## EXAMPLES. LVIII.

(1) Find log 7·65432, having given that

$$\log 7\cdot 6543 = \cdot 8839055,$$

$$\log 7\cdot 6544 = \cdot 8839112.$$

(2) Find log 564·123, having given that

$$\log 5\cdot 6412 = \cdot 7513715,$$

$$\log 5\cdot 6413 = \cdot 7513792.$$

(3) Find log ·0008736416, having given that

$$\log 8\cdot 7364 = \cdot 9413325,$$

$$\log 8\cdot 7365 = \cdot 9413375.$$

(4) Find log 6437125, having given that
log 6·4371 = ·8086903,
log 6·4372 = ·8086970.

(5) Find log 3·72456, having given that
log 37245 = 4·5710680,
log 37246 = 4·5710796.

(6) Find the number whose logarithm is ·5686760, having given that
·5686710 = log 3·7040,
·5686827 = log 3·7041.

(7) Find the number whose logarithm is 4·6602987, having given that
·6602962 = log 4·5740,
·6603057 = log 4·5741.

(8) Find the number whose logarithm is 6·3966938, having given that
·3966874 = log 2·4928,
·3967049 = log 2·4929.

(9) Find the number whose logarithm is $\bar{4}$·6431150, having given that
·6431071 = log 4·3965,
·6431170 = log 4·3966.

(10) Find the number whose logarithm is ·7550480, having given that
3·7550436 = log 5689·1,
2·7550512 = log 568·92.

225. The same Rule of Proportional Differences is used in the case of **angles** and their **Trigonometrical Ratios**; and therefore also in the case of **angles** and the **logarithms of their Ratios.**

Thus the (small) differences between three angles are assumed to be proportional to the corresponding differences between the sines of those three angles; also, proportional to the corresponding differences between the logarithms of the sines of those angles.

226. Sines and cosines are always less than unity, as also are the tangents of all angles between 0° and 45°.

The logarithms of these Ratios must therefore have *negative* characteristics.

To avoid the inconvenience of having to print these negative characteristics, the whole number 10 is added to each logarithm of the Trigonometrical Ratios, before it is set down in the Table.

The numbers thus recorded are called the **tabular logarithms** of the sine, cosine, etc., of an angle.

They are indicated by the letter 'L.'

Thus L *sin* 31° 15′, stands for the tabular logarithm of sin 31° 15′, and is equal to $\{\log(\sin 31^\circ\,15') + 10\}$.

The words **logarithmic sine** are used as abbreviation for **tabular logarithm of the sine.**

Thus in the Tables we find

$$L \sin 31^0\,15' = 9{\cdot}7149776.$$

Therefore $\log(\sin 31^0\,15') = 9{\cdot}7149776 - 10 = \bar{1}{\cdot}7149776.$

*Example* 1. *Find* sin 31° 6′ 25″.

The Tables give $\sin 31^0\,6' = {\cdot}5165333$ ........................(i),

$\sin 31^0\,7' = {\cdot}5167824$ ........................(ii).

Let $\sin 31^0\,6'\,25'' = {\cdot}5165333 + d$ ..................(iii).

Subtracting

$$\left.\begin{array}{r}{\cdot}5167824\\ \underline{{\cdot}5165333}\\ {\cdot}0002491\end{array}\right\}$$

The other difference is $d$, and the corresponding differences in the angles are 60″ and 25″;

$$\therefore\ d = \tfrac{25}{60} \text{ of } {\cdot}0002491$$
$$= {\cdot}0001038\ldots$$

Hence, from (iii) $\sin 31^0 6' 25'' = {\cdot}5165333 + {\cdot}0001038 = {\cdot}5166371.$

*Example* 2. Find the angle whose logarithmic cosine is 9·7858083.

The Table gives $9\cdot7857611 = L\cos 52^\circ 22'$ ....................(i),

$9\cdot7859249 = L\cos 52^\circ 21'$ ..................(ii).

The cosine diminishes as the angle increases. Hence corresponding to an increase in the angle there is a diminution of the cosine.

Hence, let $9\cdot7858083 = L\cos(52^\circ 22' - D)$ .................(iii).

Subtracting the first tabular logarithm from the second the difference is ·0001638.

Subtracting the first tabular logarithm from the third, the difference is ·0000472.

Subtracting the first angle from the second, the difference is $-60''$.

Subtracting the first angle from the third, the difference is $-D$.

By the Rule these four differences are in proportion.

Therefore $$D = \tfrac{472}{1638} \text{ of } 60''$$
$$= 17\cdot3''.$$

Hence $$9\cdot7858083 = L\cos(52^\circ 22' - 17'')$$
$$= L\cos 52^\circ 21' 43''$$

## EXAMPLES. LIX.

(1) Find $\sin 42^\circ 21' 30''$

having given that $\sin 42^\circ 21' = \cdot6736577$

$\sin 42^\circ 22' = \cdot6738727.$

(2) Find $\cos 47^\circ 38' 30''$

having given that $\cos 47^\circ 38' = \cdot6738727$

$\cos 47^\circ 39' = \cdot6736577.$

(3) Find $\cos 21^\circ 27' 45''$

having given that $\cos 21^\circ 27' = \cdot9307370$

$\cos 21^\circ 28' = \cdot9306306.$

(4) Find the angle whose sine is $\cdot 6666666$
having given that $\cdot 6665325 = \sin 41^\circ\ 48'$
$\cdot 6667493 = \sin 41^\circ\ 49'$.

(5) Find the angle whose cosine is $\cdot 3333333$
having given that $\cdot 3332584 = \cos 70^\circ\ 32'$
$\cdot 3335326 = \cos 70^\circ\ 31'$.

(6) Find the angle whose cosine is $\cdot 25$
having given that $\cdot 2498167 = \cos 75^\circ\ 32'$
$\cdot 2500984 = \cos 75^\circ\ 31'$.

(7) Find $L \sin 45^\circ\ 16'\ 30''$
having given that $L \sin 45^\circ\ 16' = 9{\cdot}8514969$
$L \sin 45^\circ\ 17' = 9{\cdot}8516220$.

(8) Find $L \tan 27^\circ\ 13'\ 45''$
having given that $L \tan 27^\circ\ 13' = 9{\cdot}7112148$
$L \tan 27^\circ\ 14' = 9{\cdot}7115254$.

(9) Find $L \cot 36^\circ\ 18'\ 20''$
having given that $L \cot 36^\circ\ 18' = 10{\cdot}1339650$
$L \cot 36^\circ\ 19' = 10{\cdot}1337003$.

(10) Find the angle whose Logarithmic tangent is $9{\cdot}8464028$,
having given that $9{\cdot}8463018 = L \tan 35^\circ\ 4'$
$9{\cdot}8465705 = L \tan 35^\circ\ 5'$.

(11) Find the angle whose Logarithmic cosine is $9{\cdot}9448230$,
having given that $9{\cdot}9447862 = L \cos 28^\circ\ 17'$
$9{\cdot}9448541 = L \cos 28^\circ\ 16'$.

(12) Find the angle whose Logarithmic cosecant is $10{\cdot}4274623$,
having given that $10{\cdot}4273638 = L \operatorname{cosec} 21^\circ\ 57'$
$10{\cdot}4276774 = L \operatorname{cosec} 21^\circ\ 56'$.

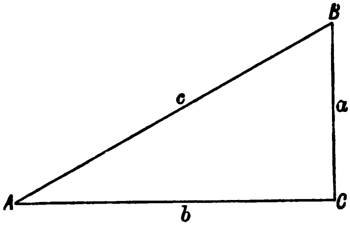
227. Problems in which each of the lines involved contains an *exact* number of feet, and each angle an *exact* number of degrees, **do not occur** in practical work.

As from time to time the skill of observers and of instrument-makers has increased, so also has the number of significant figures by which observations have been recorded.

Thus the want was felt of some method by which the labour involved in the multiplication and division of long numerical quantities could be avoided. At the end of the Seventeenth Century a celebrated Scotch mathematician, John Napier, Baron of Merchiston, proposed his method of 'Logarithms'; i.e. the method of representing numbers by **indices**; 'which, by reducing to a few days the labour of 'many months, doubles, as it were, the life of an Astronomer, 'besides freeing him from the errors and disgust inseparable 'from long calculations.' *Laplace.*

228. We shall now give a few examples of the **practical use** of logarithms.

*Example* 1. The sides containing the right angle $C$ in a right angled triangle $ABC$ contain 3456·4 ft. and 4543·5 feet respectively; find the angles of the triangle, and the length of the hypotenuse.



Let $a$, $b$, $c$ be the lengths of the sides of the triangle opposite the angles $A$, $B$, $C$ respectively.

Then, $a=3456{\cdot}4$ feet, $b=4543{\cdot}5$ feet,

$$\tan A=\frac{a}{b}.$$

Therefore $$\log\tan A=\log\frac{a}{b}=\log a-\log b$$
$$=\log 3456{\cdot}4-\log 4543{\cdot}5.$$

In the Tables we find

$$\log 3456{\cdot}4=3{\cdot}5386240$$
$$\log 4543{\cdot}5=3{\cdot}6573905.$$

$$\therefore\ \log\tan A=3{\cdot}5386240-3{\cdot}6573905$$
$$=\bar{1}{\cdot}8812335.$$

$$\therefore\ L\tan A=9{\cdot}8812335.$$

In the Table we find

$$9{\cdot}8813144=L\tan 37^\circ\,16'$$
$$9{\cdot}8810522=L\tan 37^\circ\,15'.$$

Whence by the rule of Proportional Differences

$$9{\cdot}8812335=L\tan 37^\circ\,15'\,42''.$$

Therefore
and
$$\left.\begin{aligned}A&=37^\circ\,15'\,42''\\ B=90^\circ-A&=52^\circ\,44'\,18''\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I)}.$$

To find $c$, we may take the square root of $a^2+b^2$, or we may find it by the aid of logarithms thus:

$$\frac{c}{a}=\operatorname{cosec}A=\operatorname{cosec}37^\circ\,15'\,42'',$$

$$\begin{aligned}\therefore\ \log c&=\log a+\log\operatorname{cosec}37^\circ\,15'\,42''\\ &=\log a+L\ \operatorname{cosec}37^\circ\,15'\,42''-10\\ &=3{\cdot}5386240+10{\cdot}2179174\quad-10\\ &=3{\cdot}7565414\\ &=\log 5708{\cdot}8,\end{aligned}$$

$\therefore$ the hypotenuse contains 5708·8 feet.

Thus we have found the angles and the third side of the triangle.

229. There are some formulæ which are seldom used in practical work, because they are not adapted to logarithmic calculation. They are those in which powers of quantities are connected by the signs + or –.

*Example.* In the above example we might have found the length of the hypotenuse by means of the formula

$$c^2 = a^2 + b^2.$$

But we should have had to go through the process of calculating by multiplication the values of $a^2$ and $b^2$.

For this reason, a formula which consists entirely of **factors** is always preferred to one which consists of **terms**, when any of those terms contain any power of the quantities involved.

If in the above example, the lengths of the hypotenuse $c$ and of one side $a$ were given, then the formula

$$b^2 = c^2 - a^2 = (c-a)(c+a)$$

will give the length of $b$. For

$$\log b^2 = \log \{(c-a)(c+a)\},$$

or,

$$2 \log b = \log (c-a) + \log (c+a).$$

And the values of $(c+a)$ and $(c-a)$ are easily written down from the given values of $c$ and $a$.

## EXAMPLES. LX.

In the following questions $A$, $B$, $C$ are the angles of a right angled triangle of which $C$ is a right angle, and $a$, $b$, $c$ are the lengths of the sides opposite those angles respectively.

(1) Given that $a = 1046{\cdot}7$ yards, $c = 1856{\cdot}2$ yards, $C = 90^\circ$, find $A$.

$\log 1046{\cdot}7 = 3{\cdot}0198222$, $\log 1856{\cdot}2 = 3{\cdot}2686248$,

$L \sin 34^\circ\, 19' = 9{\cdot}7510991$,

$L \sin 34^\circ\, 20' = 9{\cdot}7512842$.

(2) Given that $a = 843 \cdot 2$ feet, $C = 90^0$, and $A = 34^0\ 15'$; find $c$.

$\log 843 \cdot 2 = 2 \cdot 9259306$, $L \operatorname{cosec} 34^0\ 15' = 10 \cdot 2496421$,

$\log 1 \cdot 4982 = \cdot 17557$.

(3) Given that $a = 4845$ yards, $b = 4742$ yards, and $C = 90$, find $A$.

$\log 4845 = 3 \cdot 6852938$, $\log 4742 = 3 \cdot 6759615$,

$L \tan 45^0\ 36' = 10 \cdot 0090965$, $L \tan 45^0\ 37' = 10 \cdot 0093492$.

(4) Given that $c = 8762$ feet, $C = 90$, and $A = 37^0\ 10'$, find $a$ and $b$.

$\log 8762 = 3 \cdot 9426032$, $L \sin 37^0\ 10' = 9 \cdot 7811344$,

$L \cos 37^0\ 10' = 9 \cdot 9013938$, $\log 5 \cdot 2934 = \cdot 72373$,

$\log 6 \cdot 9823 = \cdot 843997$.

(5) Given that $b = 1694 \cdot 2$ chains, $C = 90^0$, and $A = 18^0\ 47'$, find $a$.

$\log 1694 \cdot 2 = 3 \cdot 2289647$, $L \cot 18^0\ 47' = 10 \cdot 4683893$,

$\log 5 \cdot 7620 = \cdot 76057$.

(6) Given that $a = 1072$ chains, $c = 4849$ chains, and $C = 90^0$, find $b$.

$\log 5921 = 3 \cdot 7723951$, $\log 3777 = 3 \cdot 5771470$,

$\log 4 \cdot 729 = \cdot 67477$.

(7) Given that $b = 841$ feet, $c = 3762$ feet, and $C = 90^0$, find $a$.

$\log 4603 = 3 \cdot 6630410$, $\log 2921 = 3 \cdot 4655316$,

$\log 3 \cdot 6668 = \cdot 56428$.

(8) Given that $a = 7694 \cdot 5$ chains, $b = 8471$ chains, $C = 90^0$, find $A$ and $c$.

$\log 7694 \cdot 5 = 3 \cdot 8861804$, $\log 8471 = 3 \cdot 9279347$,

$L \tan 42^0\ 15' = 9 \cdot 95824$, $L \operatorname{cosec} 42^0\ 15' = 10 \cdot 1723937$,

$\log 1 \cdot 1444 = \cdot 05857$.

230. In the following examples the student must find the necessary logarithms from the Tables.

## * MISCELLANEOUS EXAMPLES. LXI.

(1) A balloon is at a height of 2500 feet above a plain and its angle of elevation at a point in the plain is $40^{\circ} 35'$. How far is the balloon from the point of observation?

(2) A tower standing on a horizontal plain subtends an angle of $37^{\circ} 19' 30''$ at a point in the plain distant 369·5 feet from the foot of the tower. Find the height of the tower.

(3) The shadow of a tower on a horizontal plain in the sunlight is observed to be 176·23 feet and the elevation of the sun at that moment is $33^{\circ} 12'$. Find the height of the tower.

(4) From the top of a tower 163·5 feet high by the side of a river the angle of depression of a post on the opposite bank of the river is $29^{\circ} 47' 18''$. Find the distance of the post from the foot of the tower.

(5) Given $a = 673{\cdot}12$, $b = 415{\cdot}89$ chains, $C = 90^{\circ}$, find $A$ and $B$.

(6) Given $a = 576{\cdot}12$, $c = 873{\cdot}14$ chains, $C = 90^{\circ}$, find $b$ and $A$.

(7) From the top of a light-house 112·5 feet high, the angles of depression of two ships, when the line joining the ships points to the foot of the light-house, are $27^{\circ} 18'$ and $20^{\circ} 36'$ respectively. Find the distance between the ships.

(8) From the top of a cliff the angles of depression of the top and bottom of a light-house 97·25 feet high are observed to be $23^{\circ} 17'$ and $24^{\circ} 19'$ respectively. How much higher is the cliff than the light-house?

(9) Find the distance in space travelled in an hour, in consequence of the earth's rotation, by St Paul's cathedral. (Latitude of London $= 51^{\circ} 25'$, earth's diameter $= 7914$ miles.)

(10) The angle of elevation of a balloon from a station due south of it is $47^{\circ} 18' 30''$, and from another station due west of the former and distant 671·38 feet from it the elevation is $41^{\circ} 14'$. Find the height of the balloon.

On the four following pages are given a Table of the mantissæ of the logarithms of all numbers from 100 to 1000 correct to 5 significant figures.

By the aid of the principle of proportional parts this Table can be used to work correctly to 4 significant figures.

It is only in observations or measurements, made with the most elaborate care and under special circumstances, that a greater degree of accuracy is attained than that indicated by 4 significant figures [see Lock's *Arithmetic*, chapter VIII.]; hence the Table here given will be found sufficient for most numerical calculations in which Arithmetic only is involved.

*Example* i. *Find to three significant figures the length of the diagonal of a cube whose side contains* 14·7 *inches.*

Let $x$ be the number of inches in the diagonal,

then $$x^2 = 3 \times (14{\cdot}7)^2,$$

$$\therefore\ x = \sqrt{3} \times 14{\cdot}7\,;$$

$$\begin{aligned}\therefore\ \log x &= \tfrac{1}{2}\log 3 + \log 14{\cdot}7\\ &= \tfrac{1}{2}({\cdot}47712) + 1{\cdot}16732 \text{ [from the Table]}\\ &= {\cdot}23856 + 1{\cdot}16732 = 1{\cdot}40588\\ &= \log 25{\cdot}45 \text{ nearly [from the Table].}\end{aligned}$$

Hence the diagonal is 25·45... inches.

*Example* ii. *Find the value of* $(43{\cdot}72)^{\frac{5}{9}}$.

$$\left.\begin{aligned}\log 4{\cdot}37 &= {\cdot}64048\\ \log 4{\cdot}38 &= {\cdot}64147\end{aligned}\right\}\text{ difference for } {\cdot}01 = {\cdot}00099,$$

$$\therefore\ \text{difference for } {\cdot}002 = {\cdot}00020\,;$$

$$\therefore\ \log 43{\cdot}72 = 1{\cdot}64068\,;$$

$$\begin{aligned}\therefore\ \log (43{\cdot}72)^{\frac{5}{9}} &= \tfrac{5}{9} \times 1{\cdot}64068 = {\cdot}9115\ldots\\ &= \log 8{\cdot}156\ldots \text{ [from the Table].}\end{aligned}$$

Therefore $(43{\cdot}72)^{\frac{5}{9}} = 8{\cdot}156\ldots$

# TABLE OF THE LOGARITHMS OF ALL NUMBERS FROM 100 TO 1000

| No. | Log. | No. | Log. | No. | Log | No. | Log. | No. | Log. |
|---|---|---|---|---|---|---|---|---|---|
| 100 | 00000 | 143 | 15534 | 186 | 26951 | 229 | 35983 | 272 | 43457 |
| 101 | 00432 | 144 | 15836 | 187 | 27184 | 230 | 36172 | 273 | 43616 |
| 102 | 00860 | 145 | 16137 | 188 | 27416 | 231 | 36361 | 274 | 43775 |
| 103 | 01284 | 146 | 16435 | 189 | 27646 | 232 | 36549 | 275 | 43933 |
| 104 | 01703 | 147 | 16732 | 190 | 27875 | 233 | 36736 | 276 | 44091 |
| 105 | 02119 | 148 | 17026 | 191 | 28103 | 234 | 36922 | 277 | 44248 |
| 106 | 02531 | 149 | 17319 | 192 | 28330 | 235 | 37107 | 278 | 44404 |
| 107 | 02938 | 150 | 17609 | 193 | 28556 | 236 | 37291 | 279 | 44560 |
| 108 | 03342 | 151 | 17898 | 194 | 28780 | 237 | 37475 | 280 | 44716 |
| 109 | 03743 | 152 | 18184 | 195 | 29003 | 238 | 37658 | 281 | 44870 |
| 110 | 04139 | 153 | 18469 | 196 | 29226 | 239 | 37840 | 282 | 45025 |
| 111 | 04532 | 154 | 18752 | 197 | 29447 | 240 | 38021 | 283 | 45179 |
| 112 | 04922 | 155 | 19033 | 198 | 29667 | 241 | 38202 | 284 | 45332 |
| 113 | 05308 | 156 | 19312 | 199 | 29885 | 242 | 38382 | 285 | 45484 |
| 114 | 05690 | 157 | 19590 | 200 | 30103 | 243 | 38561 | 286 | 45637 |
| 115 | 06070 | 158 | 19866 | 201 | 30320 | 244 | 38739 | 287 | 45788 |
| 116 | 06446 | 159 | 20140 | 202 | 30535 | 245 | 38917 | 288 | 45939 |
| 117 | 06819 | 160 | 20412 | 203 | 30750 | 246 | 39094 | 289 | 46090 |
| 118 | 07188 | 161 | 20683 | 204 | 30963 | 247 | 39270 | 290 | 46240 |
| 119 | 07555 | 162 | 20951 | 205 | 31175 | 248 | 39445 | 291 | 46389 |
| 120 | 07918 | 163 | 21219 | 206 | 31387 | 249 | 39620 | 292 | 46538 |
| 121 | 08279 | 164 | 21484 | 207 | 31597 | 250 | 39795 | 293 | 46687 |
| 122 | 08636 | 165 | 21748 | 208 | 31806 | 251 | 39967 | 294 | 46835 |
| 123 | 08991 | 166 | 22011 | 209 | 32015 | 252 | 40140 | 295 | 46982 |
| 124 | 09342 | 167 | 22272 | 210 | 32222 | 253 | 40312 | 296 | 47129 |
| 125 | 09691 | 168 | 22531 | 211 | 32428 | 254 | 40483 | 297 | 47276 |
| 126 | 10037 | 169 | 22789 | 212 | 32634 | 255 | 40654 | 298 | 47422 |
| 127 | 10380 | 170 | 23045 | 213 | 32838 | 256 | 40824 | 299 | 47567 |
| 128 | 10721 | 171 | 23300 | 214 | 33041 | 257 | 40993 | 300 | 47712 |
| 129 | 11059 | 172 | 23553 | 215 | 33244 | 258 | 41162 | 301 | 47857 |
| 130 | 11394 | 173 | 23805 | 216 | 33446 | 259 | 41330 | 302 | 48001 |
| 131 | 11727 | 174 | 24055 | 217 | 33646 | 260 | 41497 | 303 | 48144 |
| 132 | 12057 | 175 | 24304 | 218 | 33846 | 261 | 41664 | 304 | 48287 |
| 133 | 12385 | 176 | 24551 | 219 | 34044 | 262 | 41830 | 305 | 48430 |
| 134 | 12710 | 177 | 24797 | 220 | 34242 | 263 | 41996 | 306 | 48572 |
| 135 | 13033 | 178 | 25042 | 221 | 34439 | 264 | 42160 | 307 | 48714 |
| 136 | 13354 | 179 | 25285 | 222 | 34635 | 265 | 42325 | 308 | 48855 |
| 137 | 13672 | 180 | 25527 | 223 | 34830 | 266 | 42488 | 309 | 48996 |
| 138 | 13988 | 181 | 25768 | 224 | 35025 | 267 | 42651 | 310 | 49136 |
| 139 | 14301 | 182 | 26007 | 225 | 35218 | 268 | 42813 | 311 | 49276 |
| 140 | 14613 | 183 | 26245 | 226 | 35411 | 269 | 42975 | 312 | 49415 |
| 141 | 14921 | 184 | 26482 | 227 | 35602 | 270 | 43136 | 313 | 49554 |
| 142 | 15229 | 185 | 26717 | 228 | 35793 | 271 | 43297 | 314 | 49693 |

| No. | Log. | No. | Log. | No. | Log. | No. | Log. | No. | Log. |
|---|---|---|---|---|---|---|---|---|---|
| 315 | 49831 | 361 | 55751 | 407 | 60959 | 453 | 65610 | 499 | 69810 |
| 316 | 49969 | 362 | 55871 | 408 | 61066 | 454 | 65705 | 500 | 69897 |
| 317 | 50106 | 363 | 55991 | 409 | 61172 | 455 | 65801 | 501 | 69984 |
| 318 | 50243 | 364 | 56110 | 410 | 61278 | 456 | 65896 | 502 | 70070 |
| 319 | 50379 | 365 | 56229 | 411 | 61384 | 457 | 65992 | 503 | 70157 |
| 320 | 50515 | 366 | 56348 | 412 | 61470 | 458 | 66087 | 504 | 70243 |
| 321 | 50651 | 367 | 56467 | 413 | 61595 | 459 | 66181 | 505 | 70329 |
| 322 | 50786 | 368 | 56585 | 414 | 61700 | 460 | 66276 | 506 | 70415 |
| 323 | 50920 | 369 | 56703 | 415 | 61805 | 461 | 66370 | 507 | 70501 |
| 324 | 51055 | 370 | 56820 | 416 | 61909 | 462 | 66464 | 508 | 70586 |
| 325 | 51188 | 371 | 56937 | 417 | 62014 | 463 | 66558 | 509 | 70672 |
| 326 | 51322 | 372 | 57054 | 418 | 62118 | 464 | 66652 | 510 | 70757 |
| 327 | 51455 | 373 | 57171 | 419 | 62221 | 465 | 66745 | 511 | 70842 |
| 328 | 51587 | 374 | 57287 | 420 | 62325 | 466 | 66839 | 512 | 70927 |
| 329 | 51720 | 375 | 57403 | 421 | 62428 | 467 | 66932 | 513 | 71011 |
| 330 | 51851 | 376 | 57519 | 422 | 62531 | 468 | 67025 | 514 | 71096 |
| 331 | 51983 | 377 | 57634 | 423 | 62634 | 469 | 67117 | 515 | 71181 |
| 332 | 52114 | 378 | 57749 | 424 | 62737 | 470 | 67210 | 516 | 71265 |
| 333 | 52244 | 379 | 57864 | 425 | 62839 | 471 | 67302 | 517 | 71349 |
| 334 | 52375 | 380 | 57978 | 426 | 62941 | 472 | 67394 | 518 | 71433 |
| 335 | 52504 | 381 | 58093 | 427 | 63043 | 473 | 67486 | 519 | 71517 |
| 336 | 52634 | 382 | 58206 | 428 | 63144 | 474 | 67578 | 520 | 71600 |
| 337 | 52763 | 383 | 58320 | 429 | 63246 | 475 | 67669 | 521 | 71684 |
| 338 | 52892 | 384 | 58433 | 430 | 63347 | 476 | 67761 | 522 | 71767 |
| 339 | 53020 | 385 | 58546 | 431 | 63448 | 477 | 67852 | 523 | 71850 |
| 340 | 53148 | 386 | 58659 | 432 | 63548 | 478 | 67943 | 524 | 71933 |
| 341 | 53275 | 387 | 58771 | 433 | 63649 | 479 | 68034 | 525 | 72016 |
| 342 | 53403 | 388 | 58883 | 434 | 63749 | 480 | 68124 | 526 | 72099 |
| 343 | 53529 | 389 | 58995 | 435 | 63849 | 481 | 68215 | 527 | 72181 |
| 344 | 53656 | 390 | 59106 | 436 | 63949 | 482 | 68305 | 528 | 72263 |
| 345 | 53782 | 391 | 59218 | 437 | 64048 | 483 | 68395 | 529 | 72346 |
| 346 | 53908 | 392 | 59329 | 438 | 64147 | 484 | 68485 | 530 | 72428 |
| 347 | 54033 | 393 | 59439 | 439 | 64246 | 485 | 68574 | 531 | 72509 |
| 348 | 54158 | 394 | 59550 | 440 | 64345 | 486 | 68664 | 532 | 72591 |
| 349 | 54283 | 395 | 59660 | 441 | 64444 | 487 | 68753 | 533 | 72673 |
| 350 | 54407 | 396 | 59770 | 442 | 64542 | 488 | 68842 | 534 | 72754 |
| 351 | 54531 | 397 | 59879 | 443 | 64640 | 489 | 68931 | 535 | 72835 |
| 352 | 54654 | 398 | 59988 | 444 | 64738 | 490 | 69020 | 536 | 72916 |
| 353 | 54777 | 399 | 60097 | 445 | 64836 | 491 | 69108 | 537 | 72997 |
| 354 | 54900 | 400 | 60206 | 446 | 64933 | 492 | 69197 | 538 | 73078 |
| 355 | 55023 | 401 | 60314 | 447 | 65031 | 493 | 69285 | 539 | 73159 |
| 356 | 55145 | 402 | 60423 | 448 | 65128 | 494 | 69373 | 540 | 73239 |
| 357 | 55267 | 403 | 60530 | 449 | 65225 | 495 | 69461 | 541 | 73320 |
| 358 | 55388 | 404 | 60638 | 450 | 65321 | 496 | 69548 | 542 | 73400 |
| 359 | 55509 | 405 | 60746 | 451 | 65418 | 497 | 69636 | 543 | 73480 |
| 360 | 55630 | 406 | 60853 | 452 | 65514 | 498 | 69723 | 544 | 73560 |

| No. | Log. | No. | Log. | No. | Log. | No. | Log. | No. | Log. |
|---|---|---|---|---|---|---|---|---|---|
| 545 | 73640 | 591 | 77159 | 637 | 80414 | 683 | 83442 | 729 | 86273 |
| 546 | 73719 | 592 | 77232 | 638 | 80482 | 684 | 83506 | 730 | 86332 |
| 547 | 73799 | 593 | 77305 | 639 | 80550 | 685 | 83569 | 731 | 86392 |
| 548 | 73878 | 594 | 77379 | 640 | 80618 | 686 | 83632 | 732 | 86451 |
| 549 | 73957 | 595 | 77452 | 641 | 80686 | 687 | 83696 | 733 | 86510 |
| 550 | 74036 | 596 | 77525 | 642 | 80754 | 688 | 83759 | 734 | 86570 |
| 551 | 74115 | 597 | 77597 | 643 | 80821 | 689 | 83822 | 735 | 86629 |
| 552 | 74194 | 598 | 77670 | 644 | 80889 | 690 | 83885 | 736 | 86688 |
| 553 | 74273 | 599 | 77743 | 645 | 80956 | 691 | 83948 | 737 | 86747 |
| 554 | 74351 | 600 | 77815 | 646 | 81023 | 692 | 84011 | 738 | 86806 |
| 555 | 74429 | 601 | 77887 | 647 | 81090 | 693 | 84073 | 739 | 86864 |
| 556 | 74507 | 602 | 77960 | 648 | 81158 | 694 | 84136 | 740 | 86923 |
| 557 | 74586 | 603 | 78032 | 649 | 81224 | 695 | 84198 | 741 | 86982 |
| 558 | 74663 | 604 | 78104 | 650 | 81291 | 696 | 84261 | 742 | 87040 |
| 559 | 74741 | 605 | 78176 | 651 | 81358 | 697 | 84323 | 743 | 87099 |
| 560 | 74819 | 606 | 78247 | 652 | 81425 | 698 | 84385 | 744 | 87157 |
| 561 | 74896 | 607 | 78319 | 653 | 81491 | 699 | 84448 | 745 | 87216 |
| 562 | 74974 | 608 | 78390 | 654 | 81558 | 700 | 84510 | 746 | 87274 |
| 563 | 75051 | 609 | 78462 | 655 | 81624 | 701 | 84572 | 747 | 87332 |
| 564 | 75128 | 610 | 78533 | 656 | 81690 | 702 | 84634 | 748 | 87390 |
| 565 | 75205 | 611 | 78604 | 657 | 81757 | 703 | 84696 | 749 | 87448 |
| 566 | 75281 | 612 | 78675 | 658 | 81823 | 704 | 84757 | 750 | 87506 |
| 567 | 75358 | 613 | 78746 | 659 | 81889 | 705 | 84819 | 751 | 87564 |
| 568 | 75435 | 614 | 78817 | 660 | 81954 | 706 | 84880 | 752 | 87622 |
| 569 | 75511 | 615 | 78888 | 661 | 82020 | 707 | 84942 | 753 | 87680 |
| 570 | 75587 | 616 | 78958 | 662 | 82086 | 708 | 85003 | 754 | 87737 |
| 571 | 75664 | 617 | 79029 | 663 | 82151 | 709 | 85065 | 755 | 87795 |
| 572 | 75740 | 618 | 79099 | 664 | 82217 | 710 | 85126 | 756 | 87852 |
| 573 | 75815 | 619 | 79169 | 665 | 82282 | 711 | 85187 | 757 | 87910 |
| 574 | 75891 | 620 | 79239 | 666 | 82347 | 712 | 85248 | 758 | 87967 |
| 575 | 75967 | 621 | 79309 | 667 | 82413 | 713 | 85309 | 759 | 88024 |
| 576 | 76042 | 622 | 79379 | 668 | 82478 | 714 | 85370 | 760 | 88081 |
| 577 | 76118 | 623 | 79449 | 669 | 82543 | 715 | 85431 | 761 | 88138 |
| 578 | 76192 | 624 | 79518 | 670 | 82607 | 716 | 85491 | 762 | 88196 |
| 579 | 76268 | 625 | 79588 | 671 | 82672 | 717 | 85552 | 763 | 88252 |
| 580 | 76343 | 626 | 79657 | 672 | 82737 | 718 | 85612 | 764 | 88309 |
| 581 | 76418 | 627 | 79727 | 673 | 82802 | 719 | 85673 | 765 | 88366 |
| 582 | 76492 | 628 | 79796 | 674 | 82866 | 720 | 85733 | 766 | 88423 |
| 583 | 76567 | 629 | 79865 | 675 | 82930 | 721 | 85794 | 767 | 88480 |
| 584 | 76641 | 630 | 79934 | 676 | 82995 | 722 | 85854 | 768 | 88536 |
| 585 | 76716 | 631 | 80003 | 677 | 83059 | 723 | 85914 | 769 | 88593 |
| 586 | 76790 | 632 | 80072 | 678 | 83123 | 724 | 85974 | 770 | 88649 |
| 587 | 76864 | 633 | 80140 | 679 | 83189 | 725 | 86034 | 771 | 88705 |
| 588 | 76938 | 634 | 80209 | 680 | 83251 | 726 | 86094 | 772 | 88762 |
| 589 | 77012 | 635 | 80277 | 681 | 83315 | 727 | 86153 | 773 | 88818 |
| 590 | 77085 | 636 | 80346 | 682 | 83378 | 728 | 86213 | 774 | 88874 |

| No. | Log. | No. | Log. | No. | Log. | No. | Log. | No. | Log. |
|---|---|---|---|---|---|---|---|---|---|
| 775 | 88930 | 820 | 91381 | 865 | 93702 | 910 | 95904 | 955 | 98000 |
| 776 | 88986 | 821 | 91434 | 866 | 93752 | 911 | 95952 | 956 | 98046 |
| 777 | 88042 | 822 | 91487 | 867 | 93802 | 912 | 95999 | 957 | 98091 |
| 778 | 89098 | 823 | 91540 | 868 | 93852 | 913 | 96047 | 958 | 98137 |
| 779 | 89154 | 824 | 91593 | 869 | 93902 | 914 | 96095 | 959 | 98182 |
| 780 | 89209 | 825 | 91645 | 870 | 93952 | 915 | 96142 | 960 | 98227 |
| 781 | 89265 | 826 | 91698 | 871 | 94002 | 916 | 96190 | 961 | 98272 |
| 782 | 89321 | 827 | 91751 | 872 | 94052 | 917 | 96237 | 962 | 98318 |
| 783 | 89376 | 828 | 91803 | 873 | 94101 | 918 | 96284 | 963 | 98363 |
| 784 | 89432 | 829 | 91855 | 874 | 94151 | 919 | 96332 | 964 | 98408 |
| 785 | 89487 | 830 | 91908 | 875 | 94201 | 920 | 96379 | 965 | 98453 |
| 786 | 89542 | 831 | 91960 | 876 | 94250 | 921 | 96426 | 966 | 98498 |
| 787 | 89597 | 832 | 92012 | 877 | 94300 | 922 | 96473 | 967 | 98543 |
| 788 | 89653 | 833 | 92065 | 878 | 94349 | 923 | 96520 | 968 | 98588 |
| 789 | 89708 | 834 | 92117 | 879 | 94399 | 924 | 96567 | 969 | 98632 |
| 790 | 89763 | 835 | 92169 | 880 | 94448 | 925 | 96614 | 970 | 98677 |
| 791 | 89818 | 836 | 92221 | 881 | 94498 | 926 | 96661 | 971 | 98722 |
| 792 | 89873 | 837 | 92273 | 882 | 94547 | 927 | 96708 | 972 | 98767 |
| 793 | 89927 | 838 | 92324 | 883 | 94596 | 928 | 96754 | 973 | 98811 |
| 794 | 89982 | 839 | 92376 | 884 | 94645 | 929 | 96801 | 974 | 98856 |
| 795 | 90037 | 840 | 92428 | 885 | 94694 | 930 | 96848 | 975 | 98900 |
| 796 | 90091 | 841 | 92480 | 886 | 94743 | 931 | 96895 | 976 | 98945 |
| 797 | 90146 | 842 | 92531 | 887 | 94792 | 932 | 96941 | 977 | 98990 |
| 798 | 90200 | 843 | 92583 | 888 | 94841 | 933 | 96988 | 978 | 99034 |
| 799 | 90255 | 844 | 92634 | 889 | 94890 | 934 | 97035 | 979 | 99078 |
| 800 | 90309 | 845 | 92686 | 890 | 94949 | 935 | 97081 | 980 | 99123 |
| 801 | 90363 | 846 | 92737 | 891 | 94988 | 936 | 97128 | 981 | 99167 |
| 802 | 90417 | 847 | 92788 | 892 | 95036 | 937 | 97174 | 982 | 99211 |
| 803 | 90472 | 848 | 92840 | 893 | 95085 | 938 | 97220 | 983 | 99255 |
| 804 | 90526 | 849 | 92891 | 894 | 95134 | 939 | 97267 | 984 | 99300 |
| 805 | 90580 | 850 | 92942 | 895 | 95182 | 940 | 97313 | 985 | 99344 |
| 806 | 90634 | 851 | 92993 | 896 | 95231 | 941 | 97359 | 986 | 99388 |
| 807 | 90687 | 852 | 93044 | 897 | 95279 | 942 | 97405 | 987 | 99432 |
| 808 | 90741 | 853 | 93095 | 898 | 95328 | 943 | 97451 | 988 | 99476 |
| 809 | 90795 | 854 | 93146 | 899 | 95376 | 944 | 97497 | 989 | 99520 |
| 810 | 90849 | 855 | 93197 | 900 | 95424 | 945 | 97543 | 990 | 99564 |
| 811 | 90902 | 856 | 93247 | 901 | 95472 | 946 | 97589 | 991 | 99607 |
| 812 | 90956 | 857 | 93298 | 902 | 95521 | 947 | 97635 | 992 | 99651 |
| 813 | 91009 | 858 | 93349 | 903 | 95569 | 948 | 97681 | 993 | 99695 |
| 814 | 91062 | 859 | 93399 | 904 | 95617 | 949 | 97727 | 994 | 99739 |
| 815 | 91116 | 860 | 93450 | 905 | 95665 | 950 | 97772 | 995 | 99782 |
| 816 | 91169 | 861 | 93500 | 906 | 95713 | 951 | 97818 | 996 | 99826 |
| 817 | 91222 | 862 | 93551 | 907 | 95761 | 952 | 97864 | 997 | 99870 |
| 818 | 91275 | 863 | 93601 | 908 | 95809 | 953 | 97909 | 998 | 99913 |
| 819 | 91328 | 864 | 93651 | 909 | 95856 | 954 | 97955 | 999 | 99957 |

## EXAMPLES. LXI. a.

Prove that the following statements are correct to four significant figures :

(1) $\sqrt[3]{451}=7{\cdot}669$. (2) $\sqrt[5]{802}=3{\cdot}809$.

(3) $(273)^{\frac{4}{9}}\times(234)^{\frac{1}{4}}=47{\cdot}32$. (4) $(451)^{\frac{3}{5}}\times(231)^{\frac{4}{3}}=55460$.

(5) $\left(\frac{192{\cdot}5}{84}\right)^3=12{\cdot}03$. (6) $\frac{(34{\cdot}79)^{\frac{2}{3}}}{(41{\cdot}25)^{\frac{3}{2}}}={\cdot}04023$.

(7) $\frac{(24{\cdot}76)^{\frac{2}{7}}}{({\cdot}0045)^{\frac{3}{2}}}=8287$. (8) $\frac{7{\cdot}89}{{\cdot}0345}\times(89130)^{\frac{1}{7}}=1165$.

(9) $\frac{\frac{3}{2}\sqrt{(5{\cdot}2)}}{5\sqrt{(11{\cdot}31)}}\times(\tfrac{3}{7})^{-\frac{1}{2}}={\cdot}3107$. (10) $\sqrt[5]{\left\{\frac{2\sqrt{(34)}}{3\sqrt{(791)}}\right\}}={\cdot}6731$.

(11) $\frac{\sqrt[4]{3}}{\sqrt[6]{3}}=1{\cdot}096$. (12) $\left(\frac{21^3\times 45^5}{2^7\times 3^9}\right)^{\frac{1}{2}}=823{\cdot}6$.

Solve the equations correct to 4 figures:

(13) $10^x=421$ [$x=2{\cdot}624$]. (14) $(\frac{21}{20})^x=3$ [$x=22{\cdot}51$].
(15) $(\frac{203}{200})^{2x}=2$ [$x=23{\cdot}28$]. (16) $(\frac{26}{25})^x=3$ [$x=28{\cdot}01$].
(17) $\log 37^{x+3}=3{\cdot}412$ [$x=-{\cdot}8243$].
(18) $x=10\sqrt[3]{(31{\cdot}2)}$ [$x=31{\cdot}48$].

*Example* i. *Find the* amount at Compound Interest *on* £1 *for* 8 years *at* 5 per cent.

To find the amount for 1 year we multiply by $\frac{105}{100}$, i.e. by $\frac{21}{20}$.

The amount for 2 years will be £$\frac{21}{20}\times\frac{21}{20}$ and the amount for 8 years $=(\frac{21}{20})^8$.

Let $x$ be the required amount in pounds, then

$$x=(\tfrac{21}{20})^8,$$

$$\begin{aligned}\therefore\ \log x&=8(\log 21-\log 20)\\&=8(1{\cdot}32222-1{\cdot}30103)=8({\cdot}02119)\\&={\cdot}16852=\log 1{\cdot}474\ldots\end{aligned}$$

Hence, to find the amount at Compound Interest for 8 years at 5 per cent. we multiply the Principal expressed in pounds by $1{\cdot}474+\ldots$

*Example* ii. *In how many years will the Principal be doubled at* 5 per cent. *Compound Interest?*

Let $x$ be the number of years, then

$(\frac{21}{20})^x$ is the amount at the end of $x$ years,

hence $(\frac{21}{20})^x = 2,$

or $x(\log 21 - \log 20) = \log 2, \;\therefore\; x = \frac{\cdot 30103}{\cdot 02119} = \frac{30103}{2119} = 14\cdot 2.$

*Example* iii. *Find the present value of an annuity of* £100 a year *payable yearly for* 10 years, *the first payment to be at the end of a year, Compound Interest being reckoned at* 4 *per cent. per annum.*

The principal which will amount to £1 at the end of $n$ years at 4 per cent. Comp. Int. is [See Lock's *Arithmetic*, p. 193, Ex. iv.]

$$£(\tfrac{100}{104})^n.$$

Hence the required present value is

$$£100\{r + r^2 + r^3 + \ldots + r^{10}\}, \text{ where } r = \tfrac{100}{104}$$

$$= £100\frac{1 - r^{11}}{1 - r} - £100 = £100\{1 - (\tfrac{100}{104})^{11}\} \times 26 - £100.$$

$$\log(\tfrac{100}{104})^{11} = 11(\log 100 - \log 104) = (2 - 2\cdot 01703) \times 11$$

$$= (\bar{1}\cdot 98297) \times 11 = \bar{1}\cdot 81267$$

$$= \log \cdot 6496\ldots$$

$\therefore$ the required present value $= £100 \times \cdot 3503\ldots \times 26 - £100,$

that is £810 about.

## EXAMPLES. LXI. b.

(1) Find the Compound Interest on £100 for 10 years at 4 per cent. [*Result*, £48.]

(2) Find the Compound Interest on £1 for 8 years at 5 per cent. [*Result*, 9*s.* 6½*d.*]

(3) In how many years will a sum of money be doubled at 3 per cent. Compound Interest? [*Result*, 23·4.]

(4) In how many years will a sum of money be doubled at 4 per cent. Compound Interest? [*Result*, 17·7.]

(5) Find the present value of £100 to be paid 8 years hence reckoning Compound Interest at 4 per cent. [*Result*, £73·07.]

(6) If the number of births in a town are 25 per 1000 and the deaths 20 per 1000 annually, in how many years will the population be doubled? [*After* 140 *years.*]

(7) On the birth of an infant £1000 is invested at Compound Interest in the Funds (3 per cent. payable half-yearly); calculate what it will be worth when the child is 21 years old. [*Result*, £1869.]

(8) In what time will a sum of money treble itself at 3 per cent. Compound Interest payable half-yearly? [*Result*, 36·9 *years.*]

(9) A sum of 1 shilling lent on condition of 1 penny interest being paid monthly, accumulates at Compound Interest at the same rate for 12 years; what will be then the amount? [*Result*, *about* £5066.]

(10) A man puts by 2*d.* at the end of the second week of the year, 4*d.* at the end of the fourth week, 8*d.* at the end of the sixth week; what sum would be put by for the last fortnight in the year? [*About* 67,100,000 *pence.*]

(11) A train starting from rest has at the end of 1 second velocity ·001 ft. per sec. and at the end of each second its velocity is greater by one-third than at the end of the preceding second; find the velocity in miles per hour at the end of 25 seconds. [·679 *miles per hour.*]

(12) The volume of a sphere is $\frac{4}{3}\pi\times$(cube of the radius); find the diameter of the sphere which contains a cubic yard. [*Result*, 1·24 *yards.*]

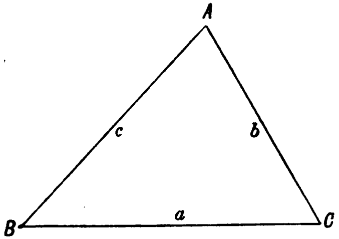
(13) Find the present value at 4 per cent. per annum Compound Interest of a Fellowship of £250 a year for six years, payable half-yearly, the first payment being due in six months. [*Result*, £1322.]

# CHAPTER XVI. ✓

## ON THE RELATIONS BETWEEN THE SIDES AND ANGLES OF A TRIANGLE.

231. The **three sides** and the **three angles** of any **triangle,** are called its six **parts.**

By the letters $A$, $B$, $C$ we shall indicate *geometrically*, the three angular points of the triangle $ABC$; *algebraically*, the three angles at those angular points respectively.



By the letters $a$, $b$, $c$ we shall indicate the measures of the sides $BC$, $CA$, $AB$ opposite the angles $A$, $B$, $C$ respectively.

✓ 232. I. We know that, $A+B+C=180^\circ$. [Euc. I. 32.]

233. Also if $A$ be an angle of a triangle, then $A$ may have any value between $0^\circ$ and $180^\circ$. Hence,

(i) **sin A** must be positive (and less than 1),

(ii) **cos A** may be positive or negative (but must be numerically less than 1),

(iii) **tan A** may have any value whatever, positive or negative.

234. Also, if we are given the value of

(i) **sin A**, there are **two** angles, each less than 180°, which have the given positive value for their sine.

(ii) **cos A**, or (iii) **tan A**, then there is only **one** value of $A$, which value can be found from the Tables.

235. $\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=90^\circ$. Therefore $\frac{A}{2}$ is less than 90°, and its Trigonometrical Ratios are all positive. Also, $\frac{A}{2}$ is known, when the value of any one of its Ratios is given. Similar remarks of course apply to the angles $B$ and $C$.

*Example* 1. *To prove* $\sin(A+B)=\sin C$.

$$A+B+C=180^\circ \therefore A+B=180^\circ-C,$$

and $\therefore$ $\sin(A+B)=\sin(180^\circ-C)=\sin C$. [p. 104.]

*Example* 2. *To prove* $\sin\frac{A+B}{2}=\cos\frac{C}{2}$,

Now $\frac{A+B+C}{2}=90^\circ$. $\therefore \frac{A+B}{2}=90^\circ-\frac{C}{2}$,

and $\therefore$ $\sin\frac{A+B}{2}=\sin\left(90^\circ-\frac{C}{2}\right)=\cos\frac{C}{2}$. [Art. 118.]

*Example* 3. *To prove*

$$\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}.$$

Now $\sin A+\sin B=2\sin\frac{A+B}{2}\cdot\cos\frac{A-B}{2}$. [Art. 157.]

$=2\cos\frac{C}{2}\cdot\cos\frac{A-B}{2}$. [Art. 118.]

and $\sin C=2\sin\frac{C}{2}\cdot\cos\frac{C}{2}$. [Art. 166.]

$=2\cos\frac{A+B}{2}\cdot\cos\frac{C}{2}$. [Art. 118.]

$$\therefore \sin A+\sin B+\sin C=2\cos\frac{C}{2}\cdot\cos\frac{A-B}{2}+2\cos\frac{C}{2}\cdot\cos\frac{A+B}{2},$$

$$=2\cos\frac{C}{2}\left(\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right).$$

$$=2\cos\frac{C}{2}\left(2\cos\frac{A}{2}\cdot\cos\frac{B}{2}\right). \qquad \text{[Art. 157.]}$$

$$=4\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}. \qquad \text{Q.E.D.}$$

## EXAMPLES. LXII.

Find $A$ from each of the six following equations, $A$ being an angle of a triangle.

(1) $\cos A=\frac{1}{2}$. (2) $\cos A=-\frac{1}{2}$. (3) $\sin A=\frac{1}{2}$.

(4) $\tan A=-1$. (5) $\sin A=\frac{1}{\sqrt{2}}$. (6) $\tan A=-\sqrt{3}$.

Prove the following statements, $A$, $B$, $C$ being the angles of a triangle.

(7) $\sin(A+B+C)=0$. (8) $\cos(A+B+C)=-1$.

(9) $\sin\frac{A+B+C}{2}=1$. (10) $\cos\frac{A+B+C}{2}=0$.

(11) $\tan(A+B)=-\tan C$. (12) $\cot\frac{B+C}{2}=\tan\frac{A}{2}$.

(13) $\cos(A+B)=-\cos C$. (14) $\cos(A+B-C)=-\cos 2C$.

(15) $\tan A-\cot B=\cos C\cdot\sec A\cdot\operatorname{cosec} B$.

(16) $\frac{\sin A-\sin B}{\sin A+\sin B}=\tan\frac{C}{2}\cdot\tan\frac{A-B}{2}$.

(17) $\frac{\sin 3B-\sin 3C}{\cos 3C-\cos 3B}=\tan\frac{3A}{2}$.

(18) $\sin A+\sin B-\sin C=4\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\cos\frac{C}{2}$.

*(19) $\sin A-\sin B+\sin C=4\sin\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\sin\frac{C}{2}$.

(20) $\sin\frac{A}{2}\cdot\cos\frac{A}{2}+\sin\frac{B}{2}\cdot\cos\frac{B}{2}+\sin\frac{C}{2}\cdot\cos\frac{C}{2}$
$=2\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}$.

(21) $\cos A+\cos B+\cos C-1=4\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}$.

(22) $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}-\cos^2\frac{C}{2}=2\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\sin\frac{C}{2}$.

(23) $\sin^2\frac{A}{2}-\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\cos\frac{C}{2}$.

(24) $\sin\frac{B+C-A}{2}+\sin\frac{C+A-B}{2}+\sin\frac{A+B-C}{2}-1$
$=4\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}$.

(25) $\sin 2A+\sin 2B+\sin 2C=4\sin A\cdot\sin B\cdot\sin C$.

(26) $\sin A\cdot\cos A-\sin B\cdot\cos B+\sin C\cdot\cos C$
$=2\cos A\cdot\sin B\cdot\cos C$.

(27) $\sin(B+C-A)-\sin(C+A-B)+\sin(A+B-C)$
$=4\cos A\cdot\sin B\cdot\cos C$.

(28) $\cos 2A+\cos 2B+\cos 2C=-1-4\cos A\cdot\cos B\cdot\cos C$.

(29) $\sin^2 A-\sin^2 B+\sin^2 C=2\sin A\cdot\cos B\cdot\sin C$.

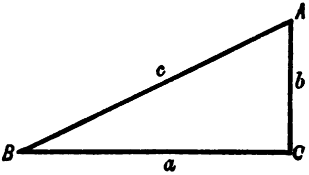
(30) $\cos(B+C-A)+\cos(C+A-B)-\cos(A+B-C)+1$
$=4\sin A\cdot\sin B\cdot\cos C$.

(31) $\sin\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}+\sin\frac{B}{2}\cdot\cos\frac{C}{2}\cdot\cos\frac{A}{2}$
$+\sin\frac{C}{2}\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}=\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}+1$.

(32) $\tan A+\tan B+\tan C=\tan A\cdot\tan B\cdot\tan C$.

(33) $\tan\frac{B}{2}\cdot\tan\frac{C}{2}+\tan\frac{C}{2}\cdot\tan\frac{A}{2}+\tan\frac{A}{2}\cdot\tan\frac{B}{2}=1$.

236. If $ABC$ be a **right-angled** triangle having $C = 90°$, then $(A + B) = 90°$.



Hence, $\sin A = \sin(90° - B) = \cos B$.

Also, $\sin A = \dfrac{a}{c} = \cos B$,

and so on. (See page 52.)

## EXAMPLES. LXIII.

In a right-angled triangle $ABC$, in which $C$ is a right angle, prove the following statements.

(1) $\tan A = \cot B$.

(2) $\tan B = \cot A + \cos C$.

(3) $\sin 2A = \sin 2B$.

(4) $\cos 2A + \cos 2B = 0$.

(5) $\sin 2A = \dfrac{2ab}{c^2}$.

(6) $\operatorname{cosec} 2B = \dfrac{a}{2b} + \dfrac{b}{2a}$.

(7) $\cos 2A = \dfrac{b^2 - a^2}{c^2}$.

(8) $\cos 2B = \dfrac{\sin^2 A - \sin^2 B}{\sin^2 A + \sin^2 B}$.

(9) $\sin^2 \dfrac{B}{2} = \dfrac{c - a}{2c}$.

(10) $\cos^2 \dfrac{A}{2} = \dfrac{c + b}{2c}$.

(11) $\left(\cos\dfrac{A}{2} + \sin\dfrac{A}{2}\right)^2 = \dfrac{a + c}{c}$.

(12) $\dfrac{a - b}{a + b} = \tan\dfrac{A - B}{2}$.

(13) $\sin(A - B) + \cos 2A = 0$.

(14) $\sin(A - B) + \sin(2A + C) = 0$.

(15) $(\sin A - \sin B)^2 + (\cos A + \cos B)^2 = 2$.

(16) $\sqrt{\dfrac{a+b}{a-b}} + \sqrt{\dfrac{a-b}{a+b}} = \dfrac{2\sin A}{\sqrt{\cos 2B}}$.

237. II. *To prove* $a = b\cos C + c\cos B$.

From $A$, any one of the angular points, draw $AD$ perpendicular to $BC$, or to $BC$ produced if necessary.

There will be three cases. Fig. i. when both $B$ and $C$ are acute angles; Fig. ii. when one of them ($B$) is obtuse; Fig. iii. when one of them ($B$) is a right angle. Then,

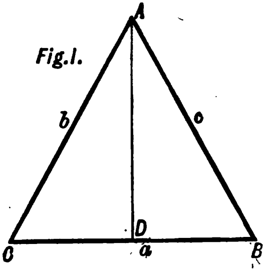


Fig. I.

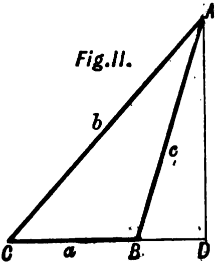


Fig. II.

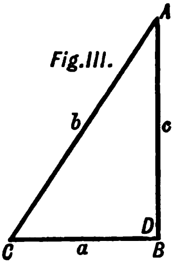


Fig. III.

**Fig. i.** $\frac{CD}{CA} = \cos ACD$; or, $CD = b\cos C$,

and $\frac{DB}{AB} = \cos ABD$; or, $DB = c\cos B$,

$\therefore a = CD + DB = b\cos C + c\cos B$.

**Fig. ii.** $\frac{CD}{CA} = \cos ACD$; or, $CD = b\cos C$,

$\frac{BD}{AB} = \cos ABD$; or, $BD = c\cos(180^\circ - B)$,

$\therefore a = CD - BD = b\cos C - c\cos(180^\circ - B)$
$= b\cos C + c\cos B$.

**Fig. iii.** $a = CB = b\cos C$
$= b\cos C + c\cos B$. [For, $\cos B = \cos 90^\circ = 0$.]

Similarly it may be proved that,

$b = c\cos A + a\cos C$; $c = a\cos B + b\cos A$.

238. III. *To prove that in any triangle, the sides are proportional to the sines of the opposite angles;* or, *To prove that* $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$.

From $A$, any one of the angular points, draw $AD$ perpendicular to $BC$, or to $BC$ produced if necessary. Then,

**Fig. i.** $AD=b\sin C$; for, $\dfrac{AD}{AC}=\sin C$ [Def.]:

$$\text{also } AD=c\sin B;\ \text{for, } \frac{AD}{AB}=\sin B.$$

$$\therefore b\sin C=c\sin B;$$

$$\text{or, } \frac{b}{\sin B}=\frac{c}{\sin C}.$$

**Fig. ii.** $AD=b\sin C$,

$$\text{and } AD=c\sin ABD=c\sin(180^\circ-B)$$

$$\therefore AD=c\sin B;$$

$$\therefore b\sin C=c\sin B;$$

$$\text{or, } \frac{b}{\sin B}=\frac{c}{\sin C}.$$

**Fig. iii.** $AB=AC\,.\,\sin C$; or, $c=b\sin C$;

$$\therefore \frac{c}{\sin C}=\frac{b}{\sin B}. \quad [\text{For } \sin B=\sin 90^\circ=1.]$$

Similarly it may be proved that

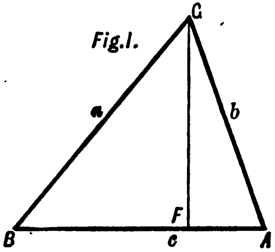
$$\frac{a}{\sin A}=\frac{b}{\sin B};$$

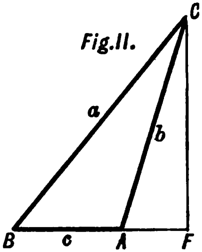
$$\therefore \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}. \qquad \text{Q.E.D.}$$

**239.** IV. *To prove that* $a^2 = b^2 + c^2 - 2bc \cos A$.

Take one of the angles $A$. Then of the other two, one must be acute. Let $B$ be an acute angle. From $C$ draw $CF$ perpendicular to $BA$, or to $BA$ produced if necessary.

There will be three figures according as $A$ is less, greater than, or equal to a right angle. Then,





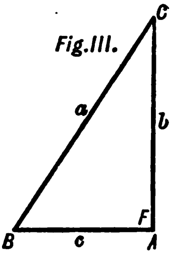


**Fig. i.** $BC^2 = CA^2 + AB^2 - 2 . BA . FA$; [Euc. II. 13]

or, $a^2 = b^2 + c^2 - 2c . FA$

$= b^2 + c^2 - 2cb \cos A$. [For $FA = b . \cos A$.]

**Fig. ii.** $BC^2 = CA^2 + AB^2 + 2 . BA . AF$; [Euc. II. 12]

or, $a^2 = b^2 + c^2 + 2cb \cos FAC$

$= b^2 + c^2 - 2bc \cos A$. [For $FAC = 180° - A$.]

**Fig. iii.** $BC^2 = CA^2 + AB^2$; [Euc. I. 47]

or, $a^2 = b^2 + c^2 - 2bc \cos A$. [For $\cos A = \cos 90° = 0$.]

Similarly it may be proved that

$$b^2 = c^2 + a^2 - 2ca \cos B,$$

and that
$$c^2 = a^2 + b^2 - 2ab \cos C.$$

**240.** V. Hence,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{c^2 + a^2 - b^2}{2ca}, \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

241. Let $s$ stand for half the sum of $a$, $b$, $c$, so that $(a+b+c)=2s$.

Then, $(b+c-a)=(b+c+a-2a)=(2s-2a)=2(s-a)$,
and $(c+a-b)=(c+a+b-2b)=(2s-2b)=2(s-b)$,
and $(a+b-c)=(a+b+c-2c)=(2s-2c)=2(s-c)$.

242. VI. *To prove that* $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$,

*and that* $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$,

*when* s *stands for half the sum of the sides* a, b, c.

**Now**, since $\cos A=\frac{b^2+c^2-a^2}{2bc}$, and $\cos A=1-2\sin^2\frac{A}{2}$; [Arts. 240, 166]

$$\therefore\ 2\sin^2\frac{A}{2}=1-\cos A=1-\frac{b^2+c^2-a^2}{2bc}$$

$$=\frac{2bc-(b^2+c^2-a^2)}{2bc}=\frac{a^2-(b^2-2bc+c^2)}{2bc}$$

$$=\frac{a^2-(b-c)^2}{2bc}=\frac{\{a-(b-c)\}\{a+(b-c)\}}{2bc}.$$

$$\therefore\ \sin^2\frac{A}{2}=\frac{(a+c-b)(a+b-c)}{4bc}.$$

$$\therefore\ \sin\frac{A}{2}=\sqrt{\frac{(2s-2b)(2s-2c)}{4bc}}=\sqrt{\frac{(s-b)(s-c)}{bc}}.$$

**Again**, since $\cos A=2\cos^2\frac{A}{2}-1$; [Art. 166]

$$\therefore\ 2\cos^2\frac{A}{2}=1+\cos A=1+\frac{b^2+c^2-a^2}{2bc}.$$

$$\therefore\ \cos^2\frac{A}{2}=\frac{(b+c)^2-a^2}{4bc}=\frac{(b+c+a)(b+c-a)}{4bc}.$$

$$\therefore\ \cos\frac{A}{2}=\sqrt{\frac{2s\,.\,(2s-2a)}{4bc}}=\sqrt{\frac{s\,.\,(s-a)}{bc}}.$$

Similarly it may be proved that

$$\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}},\quad \cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}},$$

and that

$$\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}},\quad \cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}.$$

**NOTE. In taking the square root the positive sign only is admissible before the √, because the half of an angle of a triangle is less than 90°.**

243. VII. Since $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$,

and $$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}},$$

therefore, $$\tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\frac{\sqrt{(s-b)(s-c)}}{\sqrt{s(s-a)}}.$$

244. VIII. Again,

since, $\sin A=2\sin\frac{A}{2}\cdot\cos\frac{A}{2}$; [Art. 166]

$$\therefore \sin A=2\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{s(s-a)}{bc}}$$
$$=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

The letter $S$ usually stands for $\sqrt{s(s-a)(s-b)(s-c)}$;

so that the above may be written $\frac{\sin A}{a}=\frac{2S}{abc}$.

Similarly, $$\frac{\sin B}{b}=\frac{2S}{abc}=\frac{\sin C}{c}.$$

**It should be noticed that this result gives an independent proof of III.; for**

$$\frac{2S}{abc}=\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}.$$

245. IX. *To prove that* $\dfrac{b-c}{b+c}\,.\cot\dfrac{A}{2}=\tan\dfrac{B-C}{2}$.

Since $\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$, let each of these fractions $=d$.

Then $b=d\sin B$, and $c=d\sin C$.

$$\therefore \frac{b-c}{b+c}=\frac{d\sin B-d\sin C}{d\sin B+d\sin C}=\frac{\sin B-\sin C}{\sin B+\sin C}$$

$$=\frac{2\sin\dfrac{B-C}{2}\,.\cos\dfrac{B+C}{2}}{2\sin\dfrac{B+C}{2}\,.\cos\dfrac{B-C}{2}}=\frac{\tan\dfrac{B-C}{2}}{\tan\dfrac{B+C}{2}}$$

$$=\frac{\tan\dfrac{B-C}{2}}{\cot\dfrac{A}{2}}.\qquad\left[\text{Since }\tan\frac{B+C}{2}=\tan\left(90^\circ-\frac{A}{2}\right).\right]$$

$$\therefore \frac{b-c}{b+c}\,.\cot\frac{A}{2}=\frac{\tan\dfrac{B-C}{2}}{\cot\dfrac{A}{2}}\,.\cot\frac{A}{2}=\tan\frac{B-C}{2}.\qquad\text{Q.E.D.}$$

Similarly, $\dfrac{c-a}{c+a}\,.\cot\dfrac{B}{2}=\tan\dfrac{C-A}{2}$,

and $\dfrac{a-b}{a+b}\,.\cot\dfrac{C}{2}=\tan\dfrac{A-B}{2}$.

246. NOTE. In the next chapter we shall prove, that given three of the parts of a triangle (one of which must be a side), we can find the remaining three parts by means of the formulæ of this chapter. Hence, these formulæ cannot be equivalent to more than three *independent* equations. It is an instructive, but somewhat difficult problem to take three of these formulæ (*e.g.* $A+B+C=180^\circ$, $b=c\cos A+a\cos C$, $c=b\cos A+a\cos B$) and deduce all the others from them.

**247.** The following examples are important.

*Example* 1. Suppose we are given that

$$\left.\begin{aligned} a&=b\cos C+c\cos B,\\ b&=c\cos A+a\cos C,\\ c&=a\cos B+b\cos A.\end{aligned}\right\}$$

Then, taking $a$ times the first $+b$ times the second $-c$ times the third, we get

$$a^2+b^2-c^2=(ab\cos C+ac\cos B)+(bc\cos A+ba\cos C)$$
$$-(ca\cos B+cb\cos A),$$
$$=2ab\cos C.$$

*Example* 2. Suppose we are given that $A+B+C=180^\circ$,

and that $$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}\,[=d],$$

then $$\frac{b^2+c^2-a^2}{2bc}=\frac{d^2\sin^2 B+d^2\sin^2 C-d^2\sin^2 A}{2d^2\sin B\sin C},$$

$$=\frac{\sin^2 B+\sin^2 C-\sin^2 A}{2\sin B\sin C}=\frac{2\cos A\,.\,\sin B\,.\,\sin C}{2\sin B\,.\,\sin C}$$

[Ex. 29, p. 194, since $(A+B+C)=180^\circ$.]

$$=\cos A.$$

**248.** The formula

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=d$$

is very frequently of use in solving examples.

*Example.* Prove that

$$a\cos A+b\cos B+c\cos C=2a\sin B\sin C.$$

Since $a=d\sin A,\ \ b=d\sin B,\ \ c=d\sin C,$

the above may be written

$$d\sin A\,.\,\cos A+d\sin B\,.\,\cos B+d\sin C\,.\,\cos C$$
$$=2d\sin A\,.\,\sin B\,.\,\sin C,$$

or $$\sin 2A+\sin 2B+\sin 2C=4\sin A\,.\,\sin B\,.\,\sin C,$$

which is Example (25) on page 194.

249. The student is advised to make himself thoroughly familiar with the following **formulæ**:

$$a = b\cos C + c\cos B \quad\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)},$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}\,[= d] = \frac{abc}{2S} \quad\ldots\ldots\ldots\ldots \text{(iii)},$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad\ldots\ldots\ldots\ldots\ldots\ldots \text{(v)}.$$

$$\left.\begin{aligned} \sin\frac{A}{2} &= \sqrt{\frac{(s-b)(s-c)}{bc}} \\ \cos\frac{A}{2} &= \sqrt{\frac{s(s-a)}{bc}} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots \text{(vii)},$$

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2S}{bc} \quad\ldots\ldots \text{(viii)},$$

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\,.\cot\frac{A}{2} \quad\ldots\ldots\ldots\ldots\ldots\ldots \text{(ix)}.$$

## EXAMPLES. LXIV.

In any triangle $ABC$ prove the following statements :

(1) $\dfrac{\sin A + 2\sin B}{a + 2b} = \dfrac{\sin C}{c}$.

(2) $\dfrac{\sin^2 A - m\,.\sin^2 B}{a^2 - m\,.\,b^2} = \dfrac{\sin^2 C}{c^2}$.

(3) $a\cos A + b\cos B - c\cos C = 2c\cos A\,.\cos B$.

(4) $(a+b)\sin\dfrac{C}{2} = c\cos\dfrac{A-B}{2}$.

(5) $(b-c)\cos\dfrac{A}{2} = a\sin\dfrac{B-C}{2}$.

(6) $\dfrac{\cos A}{\sin B\,.\sin C} + \dfrac{\cos B}{\sin C\,.\sin A} + \dfrac{\cos C}{\sin A\,.\sin B} = 2$.

(7) $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0.$

(8) $\dfrac{a-b}{c}=\dfrac{\cos B-\cos A}{1+\cos C}.$

(9) $\dfrac{b+c}{a}=\dfrac{\cos B+\cos C}{1-\cos A}.$

(10) $\sqrt{bc\sin B\,.\,\sin C}=\dfrac{b^2\sin C+c^2\sin B}{b+c}.$

(11) $a+b+c=(b+c)\cos A+(c+a)\cos B+(a+b)\cos C.$

(12) $b+c-a=(b+c)\cos A-(c-a)\cos B+(a-b)\cos C.$

(13) $\tan A=\dfrac{a\sin C}{b-a\cos C}.$

(14) $a(b^2+c^2)\cos A+b(c^2+a^2)\cos B+c(a^2+b^2)\cos C=3abc.$

(15) $a\cos(A+B+C)-b\cos(B+A)-c\cos(A+C)=0.$

(16) $\dfrac{\cos A}{a}+\dfrac{\cos B}{b}+\dfrac{\cos C}{c}=\dfrac{a^2+b^2+c^2}{2abc}.$

(17) $\dfrac{\tan B}{\tan C}=\dfrac{a^2+b^2-c^2}{a^2-b^2+c^2}.$ (18) $b\cos^2\dfrac{C}{2}+c\cos^2\dfrac{B}{2}=s.$

(19) $\tan\dfrac{B}{2}\,.\,\tan\dfrac{C}{2}=\dfrac{b+c-a}{b+c+a}.$

(20) $\tan\dfrac{A}{2}(b+c-a)=\tan\dfrac{B}{2}(c+a-b).$

(21) $c^2=(a+b)^2\sin^2\dfrac{C}{2}+(a-b)^2\cos^2\dfrac{C}{2}.$

## * * MISCELLANEOUS EXAMPLES. LXV.

(1) If $p$ is the length of the perpendicular from $A$ on $BC$,

$$\sin A=\frac{ap}{bc}.$$

(2) If $2\cos B\,.\,\sin C=\sin A$, then $B=C$.

(3) If $A=3B$, then $\sin B=\frac{1}{2}\sqrt{\dfrac{3b-a}{b}}.$

(4) If $\sqrt{bc \sin B \,.\, \sin C} = \dfrac{b^2 \sin B + c^2 \sin C}{b+c}$, then $B = C$.

(5) $$a \cos\frac{B}{2} \,.\, \cos\frac{C}{2} \,.\, \operatorname{cosec}\frac{A}{2} = b \cos\frac{C}{2} \,.\, \cos\frac{A}{2} \,.\, \operatorname{cosec}\frac{B}{2}$$
$$= c \cos\frac{A}{2} \,.\, \cos\frac{B}{2} \,.\, \operatorname{cosec}\frac{C}{2} = s.$$

(6) Given $\cos A = \frac{3}{5}$ and $\cos B = \frac{12}{13}$, prove that $\cos C = -\frac{16}{65}$.

(7) If $\sin^2 B + \sin^2 C = \sin^2 A$, then $A = 90^0$.

(8) If $\sin 2B + \sin 2C = \sin 2A$, then either $B = 90^0$ or $C = 90^0$.

(9) If $A : B : C = 1 : 2 : 5$, then $1 + 4 \cos A \,.\, \cos B \,.\, \cos C = 0$, and $a^2$, $b^2$, $c^2$ are in A.P.

(10) $$a \sin\frac{A}{2} \,.\, \sin\frac{B-C}{2} + b \sin\frac{B}{2} \,.\, \sin\frac{C-A}{2}$$
$$+ c \sin\frac{C}{2} \,.\, \sin\frac{A-B}{2} = 0.$$

(11) If $D$ is the middle point of $BC$, prove that
$$4AD^2 = 2b^2 + 2c^2 - a^2.$$

(12) Given that $a = 2b$, and that $A = 3B$, prove that $C = 60^0$.

(13) $abc\,(a \cos A + b \cos B + c \cos C) = 8S^2$.

(14) If $b \cos^2\dfrac{C}{2} + c \cos^2\dfrac{B}{2} = \dfrac{3a}{2}$, then $b$, $a$, $c$ are in A.P.

(15) If $D$, $E$, $F$ are the middle points of the sides $BC$, $CA$, $AB$, then
$$4\,(AD^2 + BE^2 + CF^2) = 3\,(a^2 + b^2 + c^2).$$

(16) If $D$ is the middle point of $BC$, $\cot ADB = \dfrac{b^2 - c^2}{4S}$.

(17) If $d$, $e$, $f$ are the perpendiculars from $A$, $B$, $C$ on the opposite sides of the triangle, then
$$a \sin A + b \sin B + c \sin C = 2\,(d \cos A + e \cos B + f \cos C).$$

[Some of the Examples in the Appendix might be worked by the student at this stage.]

# CHAPTER XVII.

## ON THE SOLUTION OF TRIANGLES.

250. The problem known as the **Solution of Triangles** may be stated thus: *When a sufficient number of the parts of a triangle are given, to find the magnitude of each of the other parts.*

251. When **three** parts of a Triangle (one of which must be a side) are given, the other parts can in general be determined.

There are four cases.

I. Given **three sides**. [Compare Euc. I. 8.]

II. Given **one side** and **two angles.** [Euc. I. 26.]

III. Given **two sides** and the **angle between** them. [Euc. I. 4.]

IV. Given **two sides** and the **angle opposite** one of them. [Compare Euc. VI. 7.]

### *Case* I.

252. Given **three sides**, $a$, $b$, $c$. [Euc. I. 8; VI. 5.]

We find two of the angles from the formulæ

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}.$$

The third angle $C=180-A-B$.

253. In practical work we proceed as follows:

$$\log \tan \frac{A}{2} = \log \sqrt{\frac{(s-b)(s-c)}{s(s-a)}};$$

or,

$$L \tan \frac{A}{2} - 10 = \tfrac{1}{2}\{\log (s-b) + \log (s-c) - \log s - \log (s-a)\}.$$

Similarly,

$$L \tan \frac{B}{2} - 10 = \tfrac{1}{2}\{\log (s-c) + \log (s-a) - \log s - \log (s-b)\}.$$

254. Either of the formulæ $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, $\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$ may also be used as above.

The $\sin \frac{A}{2}$ and the $\cos \frac{A}{2}$ formulæ are either of them as convenient as the $\tan \frac{A}{2}$ formula, when *one* of the angles only is to be found. If *all* the angles are to be found the tangent formula is convenient, because we can find the $L$ tangents of two half angles from the same *four* logs, viz. $\log s$, $\log (s-a)$, $\log (s-b)$, $\log (s-c)$. To find the $L$ sines of *two* half angles we require the *six* logarithms, viz. $\log (s-a)$, $\log (s-b)$, $\log (s-c)$, $\log a$, $\log b$, $\log c$.

*Example.* Given $a = 275{\cdot}35$, $b = 189{\cdot}28$, $c = 301{\cdot}47$ chains, find $A$ and $B$.

Here, $s = 383{\cdot}05$, $s - a = 107{\cdot}70$, $s - b = 193{\cdot}77$, $s - c = 81{\cdot}58$.

Then

$$\begin{aligned} L \tan \frac{A}{2} &= 10 + \tfrac{1}{2}\{\log 193{\cdot}77 + \log 81{\cdot}58 - \log 383{\cdot}05 - \log 107{\cdot}70\} \\ &= 10 + \tfrac{1}{2}\{2{\cdot}2872865 + 1{\cdot}9115837 - 2{\cdot}5832555 - 2{\cdot}0322157\} \\ &= 9{\cdot}7916995 \qquad \text{[from the tables]}, \end{aligned}$$

whence $\frac{A}{2} = 31^0\,45'\,28{\cdot}5''$; $\therefore A = 63^0\,30'\,57''$.

Also

$$L \tan \frac{B}{2} = 10 + \tfrac{1}{2} \{\log 81{\cdot}58 + \log 107{\cdot}70 - \log 383{\cdot}05 - \log 193{\cdot}77\}$$

$$= 9{\cdot}5366287 = L \tan 18^0\, 59'\, 9{\cdot}8'';$$

$$\therefore B = 37^0\, 58'\, 20''; \quad C = 180^0 - A - B = 78^0\, 30'\, 43''.$$

255. This Case may also be solved by the formula

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

But this formula is not adapted for logarithmic calculation, and therefore is seldom used in practice.

It may sometimes be used with advantage, when the given lengths of $a$, $b$, $c$ each contain less than three digits.

*Example.* Find the greatest angle of the triangle whose sides are 13, 14, 15.

Let $a=15$, $b=14$, $c=13$. Then the greatest angle is $A$.

$$\text{Now, } \cos A = \frac{14^2 + 13^2 - 15^2}{2 \times 14 \times 13} = \frac{140}{2 \times 14 \times 13} = \frac{5}{13} = {\cdot}384615.$$

$$= \cos 67^0\, 23', \text{ nearly.}$$

[By the Table of natural cosines.]

$$\therefore \text{the greatest angle} = 67^0\, 23'.$$

## EXAMPLES. LXVI.

(1) If $a = 352{\cdot}25$, $b = 513{\cdot}27$, $c = 482{\cdot}68$ yards, find the angle $A$, having given

$$\log 674{\cdot}10 = 2{\cdot}8287243, \quad \log 321{\cdot}85 = 2{\cdot}5076535,$$

$$\log 160{\cdot}83 = 2{\cdot}2063401, \quad \log 191{\cdot}42 = 2{\cdot}2819873,$$

$$L \tan 20^0\, 38' = 9{\cdot}5758104, \quad L \tan 20^0\, 39' = 9{\cdot}5761934.$$

(2) Find the two largest angles of the triangle whose sides are 484, 376, 522 chains, having given that

$$\log 6{\cdot}91 = {\cdot}8394780,\quad \log 3{\cdot}15 = {\cdot}4983106,$$
$$\log 2{\cdot}07 = {\cdot}3159703,\quad \log 1{\cdot}69 = {\cdot}2278867,$$
$$L\tan 36^\circ\, 46'\, 6'' = 9{\cdot}8734581,\quad L\tan 31^\circ\, 23'\, 9'' = 9{\cdot}7853745.$$

(3) If $a = 5238$, $b = 5662$, $c = 9384$ yards, find the angles $A$ and $B$, having given

$$\log 1{\cdot}0142 = {\cdot}0061236,\quad \log 4{\cdot}904 = {\cdot}6905505,$$
$$\log 4{\cdot}48 = {\cdot}6512780,\quad \log 7{\cdot}58 = {\cdot}8796692,$$
$$L\tan 14^\circ\, 38' = 9{\cdot}4168099,\quad L\tan 15^\circ\, 57' = 9{\cdot}4560641,$$
$$L\tan 14^\circ\, 39' = 9{\cdot}4173265,\quad L\tan 15^\circ\, 58' = 9{\cdot}4565420.$$

(4) If $a = 4090$, $b = 3850$, $c = 3811$ yards, find $A$, having given

$$\log 5{\cdot}8755 = {\cdot}7690448,\quad \log 3{\cdot}85 = {\cdot}5854607,$$
$$\log 1{\cdot}7855 = {\cdot}2517599,\quad \log 3{\cdot}811 = {\cdot}5810389,$$
$$L\cos 32^\circ\, 15' = 9{\cdot}9272306,\quad L\cos 32^\circ\, 16' = 9{\cdot}9271509.$$

(5) Find the greatest angle in a triangle whose sides are 7 feet, 8 feet, and 9 feet, having given

$$\log 3 = {\cdot}4771213,\quad L\cos 36^\circ\, 42' = 9{\cdot}9040529,$$
$$\log 1{\cdot}4 = {\cdot}146128,\quad \text{diff. for } 60'' = {\cdot}0000942.$$

(6) Find the smallest angle of the triangle whose sides are 8 feet, 10 feet, and 12 feet, having given that

$$\log 2 = {\cdot}30103,\quad L\sin 20^\circ\, 42' = 9{\cdot}5483585,\quad \text{diff. for } 60'' = {\cdot}0003342.$$

(7) If $a : b : c = 4 : 5 : 6$, find $C$, having given

$$\log 2 = {\cdot}3010300,\quad \log 3 = {\cdot}4771213,$$
$$L\cos 41^\circ\, 25' = 9{\cdot}8750142,\quad \text{diff. for } 60'' = {\cdot}0001115.$$

(8) The sides of a triangle are 2, $\sqrt{6}$, and $1 + \sqrt{3}$, find the angles.

(9) The sides of a triangle are 2, $\sqrt{2}$ and $\sqrt{3} - 1$, find the angles.

### *Case* II.

256. **Given one side and two angles**, as $a$, $B$, $C$.
[Euc. I. 26; VI. 4.]

First, $A = 180^\circ - B - C$; which determines $A$.

Next, $$\frac{b}{\sin B} = \frac{a}{\sin A}, \quad \text{or, } b = \frac{a \,.\, \sin B}{\sin A};$$

and, $$\frac{c}{\sin C} = \frac{a}{\sin A}, \quad \text{or, } c = \frac{a \,.\, \sin C}{\sin A}.$$

These determine $b$ and $c$.

257. In practical work we proceed as follows:

$$\text{Since } b = \frac{a \,.\, \sin B}{\sin A},$$

$$\therefore \log b = \log \frac{a \,.\, \sin B}{\sin A}$$

$$\therefore \log b = \log a + \log (\sin B) + 10 - (10 + \log \sin A).$$

or, $$\log b = \log a + L \sin B - L \sin A.$$

Similarly, $\log c = \log a + L \sin C - L \sin A$.

*Example.* Given that $c = 1764{\cdot}3$ feet, $C = 18^\circ 27'$, and $B = 66^\circ 39'$, find $b$.

From the Tables we find $\log 1764{\cdot}3 = 3{\cdot}2465724$.

$$L \sin 18^\circ 27' = 9{\cdot}5003421, \quad L \sin 66^\circ 39' = 9{\cdot}9628904;$$

$$\therefore \log b = 3{\cdot}2465724 + 9{\cdot}9628904 - 9{\cdot}5003421$$

$$= 3{\cdot}7091207 = \log 5118{\cdot}2;$$

$$\therefore b = 5118{\cdot}2 \text{ feet.}$$

## EXAMPLES. LXVII.

(1) If $A = 53^\circ 24'$, $B = 66^\circ 27'$, $c = 338{\cdot}65$ yards, find $C$ and $a$, having given that

$$L \sin 53^\circ 24' = 9{\cdot}9046168, \quad \log 3{\cdot}3865 = {\cdot}5297511,$$
$$L \sin 60^\circ \ 9' = 9{\cdot}9381851, \quad \log 3{\cdot}1346 = {\cdot}4961821,$$
$$\log 3{\cdot}1347 = {\cdot}4961960.$$

(2) If $A = 48^\circ$, $B = 54^\circ$, and $c = 38$ inches, find $a$ and $b$, having given that

$$\log 38 = 1{\cdot}5797836, \quad \log 2{\cdot}88704 = {\cdot}4604527,$$
$$\log 3{\cdot}14295 = {\cdot}4973368, \quad L \sin 54^\circ = 9{\cdot}9079576,$$
$$L \sin 78^\circ = 9{\cdot}9904044, \quad L \sin 48^\circ = 9{\cdot}8710735.$$

(3) Find $c$, having given that $a = 1000$ yards, $A = 50^\circ$, $C = 66^\circ$, and that

$$L \sin 50^\circ = 9{\cdot}8842540, \quad L \sin 66^\circ = 9{\cdot}9607302,$$
$$\log 1{\cdot}19255 = {\cdot}0764762.$$

(4) Find $b$, having given that $B = 32^\circ 15'$, $C = 21^\circ 47' 20''$, $a = 34$ feet.

$$\log 3{\cdot}4 = {\cdot}531479, \quad L \sin 32^\circ 15' = 9{\cdot}727228,$$
$$\log 2{\cdot}241 = {\cdot}350442, \quad L \sin 54^\circ \ 2' = 9{\cdot}908141,$$
$$\log 2{\cdot}242 = {\cdot}350636, \quad L \sin 54^\circ \ 3' = 9\ 908233.$$

(5) Find $a$, $b$, $C$, having given $A = 72^\circ 4'$, $B = 41^\circ 56' 18''$, $c = 24$ feet.

$$\log 2{\cdot}4 = {\cdot}3802112, \quad L \sin 72^\circ 4' = 9{\cdot}9783702,$$
$$\log 1{\cdot}755 = {\cdot}2442771, \quad L \sin 41^\circ 56' 10'' = 9{\cdot}8249725,$$
$$\log 1{\cdot}756 = {\cdot}2445245, \quad L \sin 41^\circ 56' 20'' = 9{\cdot}8249959,$$
$$\log 2{\cdot}4995 = {\cdot}3978531, \quad L \sin 65^\circ 59' = 9{\cdot}9606739,$$
$$\log 2{\cdot}4996 = {\cdot}3978705, \quad L \sin 66^\circ = 9{\cdot}9607302.$$

### *Case* III.

258. **Given two sides and the included angle,** as $b, c, A$.
[Euc. I. 4; VI. 6.]

First, $B + C = 180° - A$. Thus $(B + C)$ is determined.

Next, $$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}.$$

Thus $(B - C)$ is determined.

And $B$ and $C$ can be found when the values of $(B + C)$ and $(B - C)$ are known.

Lastly, $$\frac{a}{\sin A} = \frac{b}{\sin B}, \text{ or } a = \frac{b \, . \sin A}{\sin B}.$$

Whence $a$ is determined.

259. In practical work we proceed as follows;

Since $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$,

$$\therefore \log\left(\tan\frac{B-C}{2}\right) + 10$$
$$= \log(b-c) - \log(b+c) + \log\left(\cot\frac{A}{2}\right) + 10,$$

or, $L\tan\frac{B-C}{2} = \log(b-c) - \log(b+c) + L\cot\frac{A}{2}$.

Also, since $$a = \frac{b \, . \sin A}{\sin B},$$

$\therefore \log a = \log b + L\sin A - L\sin B$, as in Case II.

*Example.* Given $b = 456 \cdot 12$ chains, $c = 296 \cdot 86$ chains, and $A = 74^0\, 20'$, find the other angles.

Here, $b - c = 159 \cdot 26$, $b + c = 752 \cdot 98$.

From the Table we find,

$$\log 159 \cdot 26 = 2 \cdot 2021067, \text{ and } \log 752 \cdot 98 = 2 \cdot 8767884,$$

$$L \cot 37^0\, 10' = 10 \cdot 1202593\,;$$

$$\therefore L \tan \frac{B - C}{2} = 2 \cdot 2021067 - 2 \cdot 8767834 + 10 \cdot 1202593$$

$$= 9 \cdot 4455826 = L \tan 15^0\, 35'\, 18''.$$

$$\therefore B - C = 31^0\, 10'\, 36'', \text{ and } B + C = 180^0 - 74^0\, 20'.$$

Thus $\qquad B + C = 105^0\, 40'\,;$

$$\therefore 2B = 136^0\, 50'\, 36''\,; \quad 2C = 74^0\, 29'\, 24'',$$

or, $\qquad B = 68^0\, 25'\, 18''\,;$ or, $C = 37^0\, 14'\, 42''.$

260. The formula $a^2 = b^2 + c^2 - 2bc \cos A$ may be used in simple cases; or it may be adapted to logarithmic calculation by the use of a subsidiary angle. [Cf. Example LXXVI. (20), p. 256.]

*Example.* If $b = 35$ feet, $c = 21$ feet, and $A = 50^0$, find $a$, given that $\cos 50^0 = \cdot 643$.

Here $\qquad a^2 = 35^2 + 21^2 - 2 \times 35 \times 21 \times \cos 50^0\,;$

$$\therefore \frac{a^2}{7^2} = 5^2 + 3^2 - 2 \times 5 \times 3 \times \cos 50^0,$$

$$= 25 + 9 - 30 \times \cdot 643, \quad = 14 \cdot 71.$$

$$\therefore \frac{a}{7} = 3 \cdot 82 \text{ nearly}\,; \quad \text{or, } a = 26 \cdot 74 = \text{about } 26\tfrac{3}{4} \text{ feet.}$$

## EXAMPLES. LXVIII.

(1) Find $B$ and $C$, having given that $A = 40^0$, $b = 131$, $c = 72$.

$\log 5 \cdot 9 = \cdot 7708520$, $L \cot 20^0 = 10 \cdot 4389341$,

$\log 2 \cdot 03 = \cdot 3074960$, $L \tan 38^0\, 36' = 9 \cdot 9021604$,

$L \tan 38^0\, 37' = 9 \cdot 9024195$.

(2) Find $A$ and $B$, having given that $a=35$ feet, $b=21$ feet, $C=50^\circ$. $\log 2 = \cdot301030$, $L\tan 28^\circ\,11' = 9\cdot729020$,

$L\tan 65^\circ = 10\cdot331327$, $L\tan 28^\circ\,12' = 9\cdot729323$.

(3) If $b=19$ chains, $c=20$ chains, $A=60^\circ$, find $B$ and $C$, having given that $\log 3\cdot9 = \cdot591065$, $L\tan 2^\circ\,32' = 8\cdot645853$,

$L\cot 30^\circ = 10\cdot238561$, $L\tan 2^\circ\,33' = 8\cdot648704$.

(4) Given that $a=376\cdot375$ chains, $b=251\cdot765$ chains, and $C=78^\circ\,26'$, find $A$ and $B$. $L\cot 39^\circ\,13' = 10\cdot0882755$,

$\log 1\cdot2461 = \cdot0955529$, $L\tan 13^\circ\,39' = 9\cdot3853370$,
$\log 6\cdot2814 = \cdot7980565$, $L\tan 13^\circ\,40' = 9\cdot3858876$.

(5) If $a=135$, $b=105$, $C=60^\circ$, find $A$, having given that

$\log 2 = \cdot3010300$, $L\tan 12^\circ\,12' = 9\cdot3348711$,
$\log 3 = \cdot4771213$, $L\tan 12^\circ\,13' = 9\cdot3354823$.

(6) If $a=21$ chains, $b=20$ chains, $C=60^\circ$, find $c$.

(7) Find $c$ in the triangle of example (5).

(8) In a triangle the ratio of two sides is 5 : 3 and the included angle is $70^\circ\,30'$. Find the other angles.

$\log 2 = \cdot3010300$, $L\cot 35^\circ\,15' = 10\cdot1507464$,
$L\tan 19^\circ\,28'\,50'' = 9\cdot5486864$.

## *Case* IV.

261. Given **two sides** and the **angle opposite** one of them, as $b$, $c$, $B$. [Omitted in Euc. I; Euc. VI. 7.]

First, since $\dfrac{c}{\sin C} = \dfrac{b}{\sin B}$; $\therefore\ \sin C = \dfrac{c\sin B}{b}$.

$C$ must be found from this equation.

When $C$ is known, $A = 180^\circ - B - C$,

and, $$a = \frac{b \sin A}{\sin B}.$$

Which solves the triangle.

262. We remark however that the angle $C$, found from the **trigonometrical** equation *sin* $C =$ *a given quantity*, where $C$ is an angle of a triangle, has **two** values, one less than 90°, and one greater than 90°. [Art. 234.]

The question arises, Are both these values admissible?

This may be decided as follows:

If $B$ is not less than 90°, $C$ must be less than 90°; and the smaller value for $C$ **only** is admissible.

If $B$ is less than 90° we proceed thus.

1. If $b$ is less than $c \sin B$, then $\sin C$, which $= \dfrac{c \sin B}{b}$, is greater than 1. This is impossible. Therefore if $b$ is less than $c \sin B$, there is **no** solution whatever.

2. If $b$ is equal to $c \sin B$, then $\sin C = 1$, and therefore $C = 90^\circ$; and there is only **one** value of $C$, viz. 90°.

3. If $b$ is greater than $c \sin B$, and less than $c$, then $B$ is less than $C$, and $C$ may be obtuse or acute. In this case $C$ may have either of the values found from the equation $\sin C = \dfrac{c \sin B}{b}$. Hence there are **two** solutions, and the triangle is said to be **ambiguous.**

4. If $b$ is equal to or greater than $c$, then $B$ is equal to or greater than $C$, so that $C$ must be an acute angle; and the smaller value for $C$ **only** is admissible.

263. The same results may be obtained **geometrically**.

Construction. Draw $AB = c$; make the angle $ABD =$ the given angle $B$; with centre $A$ and radius $= b$ describe a circle; draw $AD$ perpendicular to $BD$.

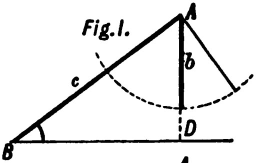


Fig. I.

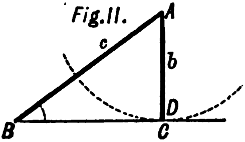


Fig. II.

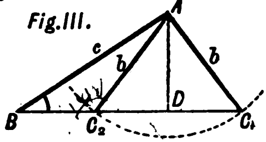


Fig. III.

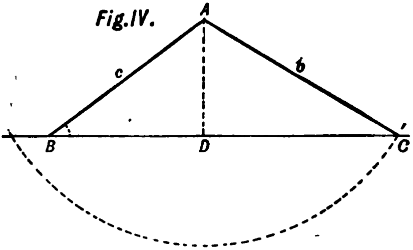


Fig. IV.

Then $AD = c \sin B$.

1. If $b$ is less than $c \sin B$, i.e. less than $AD$, the circle will not cut $BD$ at all, and the construction fails. (Fig. i.)

2. If $b$ is equal to $AD$, the circle will *touch* the line $BD$ in the point $D$, and the required triangle is the **right-angled** triangle $ABD$. (Fig. ii.)

3. If $b$ is greater than $AD$ and less than $AB$, i.e. than $c$, the circle will cut the line $BD$ in *two* points $C_1$, $C_2$ each on the same side of $B$. And we get **two** triangles $ABC_1$, $ABC_2$ each satisfying the given condition. (Fig. iii.)

4. If $b$ is equal to $c$, the circle cuts $BD$ in $B$ and in one other point $C$; if $b$ is greater than $c$ the circle cuts $BD$ in *two* points, but on *opposite* sides of $B$. In either case there is only **one** triangle satisfying the given condition. (Fig. iv.)

264. We may also obtain the same results algebraically, from the formula

$$b^2 = c^2 + a^2 - 2c\,a\cos B.$$

In this $b$, $c$, $B$ are given, $a$ is unknown. Write $x$ for $a$ and we get the quadratic equation

$$x^2 - 2c\cos B\,.\,x = b^2 - c^2.$$

Whence, $x^2 - 2c\cos B\,.\,x + c^2\cos^2 B = b^2 - c^2 + c^2\cos^2 B$

$$= b^2 - c^2\sin^2 B;$$

$$\therefore\ x = c\cos B \pm \sqrt{b^2 - c^2\sin^2 B}.$$

Let $a_1$, $a_2$ be the two values of $x$ thus obtained, then

$$\left.\begin{aligned} a_1 &= c\cos B + \sqrt{b^2 - c^2\sin^2 B} \\ a_2 &= c\cos B - \sqrt{b^2 - c^2\sin^2 B} \end{aligned}\right\}.$$

Which of these two solutions is admissible, may be decided as follows;

1. If $b$ is less than $c\sin B$, then $(b^2 - c^2\sin^2 B)$ is negative, so that $a_1$, $a_2$ are impossible quantities.

2. If $b$ is equal to $c\sin B$, then $(b^2 - c^2\sin^2 B) = 0$, and $a_1 = a_2$; thus the *two* solutions become one.

3. If $b$ is greater than $c\sin B$, then the two values $a_1$, $a_2$ are different and positive unless

$$\sqrt{b^2 - c^2\sin^2 B} \text{ is } > c\cos B,$$

i.e. unless $b^2 - c^2\sin^2 B > c^2\cos^2 B$,

i.e. unless $b^2 > c^2$.

4. If $b$ is equal to $c$, then $a_2 = 0$; if $b$ is greater than $c$, then $a_2$ is negative and is therefore inadmissible. In either of these cases $a_1$ is the only available solution.

265. We give two examples. In the first there are two solutions, in the second there is only one.

*Example* 1. Find $A$ and $C$, having given that $b=379{\cdot}41$ chains, $c=483{\cdot}74$ chains, and $B=34^{0}\,11'$.

$$\begin{aligned} L\sin C &= \log c + L\sin B - \log b \\ &= 2{\cdot}6846120 + 9{\cdot}7496148 - 2{\cdot}5791088 \\ &= 9{\cdot}8551180 = L\sin 45^{0}\,45'; \end{aligned}$$

$$\therefore\ C = 45^{0}\,45', \text{ or, } 180^{0} - 45^{0}\,45' = 134^{0}\,15'.$$

Since $b$ is less than $c$, each of these values is admissible.

When $C=\ 45^{0}\,45'$, then $A=100^{0}\,4'$.

When $C=134^{0}\,15'$, then $A=11^{0}\,34'$.

*Example* 2. Find $A$ and $C$, when $b=483{\cdot}74$ chains, $c=379{\cdot}41$ chains, and $B=34^{0}\,11'$.

$$\begin{aligned} L\sin C &= \log c + L\sin B - \log b \\ &= 2{\cdot}5791088 + 9{\cdot}7496148 - 2{\cdot}6846120 \\ &= 9{\cdot}6441116 = L\sin 26^{0}\,9'; \end{aligned}$$

$$\therefore\ C = 26^{0}\,9', \text{ or, } 180^{0} - 26^{0}\,9' = 153^{0}\,51'.$$

Since $b$ is greater than $c$, $C$ must be less than $90^{0}$, and the larger value for $C$ is inadmissible.

[It is also clear that $(153^{0}\,51' + 34^{0}\,11')$ is $> 180^{0}$].

$$\therefore\ C = 26^{0}\,9',\quad A = 119^{0}\,40'.$$

## EXAMPLES. LXIX.

(1) If $B=40^{0}$, $b=140{\cdot}5$ feet, $a=170{\cdot}6$ feet, find $A$ and $C$.

$\log 1{\cdot}405 = {\cdot}1476763$, $L\sin 40^{0} = 9{\cdot}8080675$,

$\log 1{\cdot}706 = {\cdot}2319790$, $L\sin 51^{0}\,18' = 9{\cdot}8923342$,

$L\sin 51^{0}\,19' = 9{\cdot}8924354$.

(2) Find $B$ and $C$, having given that $A=50^0$, $b=119$ chains, $a=97$ chains, and that

$$\log 1{\cdot}19={\cdot}075547,\quad L\sin 50^0=9{\cdot}884254,$$
$$\log 9{\cdot}7={\cdot}986772,\quad L\sin 70^0=9{\cdot}972986,$$
$$L\sin 70^0\,1'=9{\cdot}973032.$$

(3) Find $B$, $C$, and $c$, having given that $A=50^0$, $b=97$, $a=119$ (see example (2)).

$$\log 1{\cdot}553={\cdot}191169,\quad L\sin 38^0\,38'\,24''=9{\cdot}795479,$$
$$L\sin 88^0\,38'\,24''=9{\cdot}999876.$$

(4) Find $A$, having given that $a=24$, $c=25$, $C=65^0\,59'$, and that

$$\log 2{\cdot}5={\cdot}3979400,\quad L\sin 65^0\,59'=9{\cdot}9606739,$$
$$\log 2{\cdot}4={\cdot}3802112,\quad L\sin 61^0\,16'=9{\cdot}9429335,$$
$$L\sin 61^0\,17'=9{\cdot}9430028.$$

(5) If $a=25$, $c=24$, and $C=65^0\,59'$, find $A$, $B$ and the greater value of $b$.

$$\log 1{\cdot}755={\cdot}2442771,\quad L\sin 72^0\,4'=9{\cdot}9783702,$$
$$\log 1{\cdot}756={\cdot}2445245,\quad L\sin 72^0\,5'=9{\cdot}9784111,$$
$$L\sin 41^0\,56'\,10''=9{\cdot}8249725,$$
$$L\sin 41^0\,56'\,26''=9{\cdot}8249959;$$

for other logs see example (4).

(6) Supposing the data for the solution of a triangle to be as in the three following cases ($\alpha$), ($\beta$), ($\gamma$), point out whether the solution will be ambiguous or not, and find the third side in the obtuse-angled triangle in the ambiguous case :

($\alpha$) $A=30^0$, $a=125$ feet, $c=250$ feet,
($\beta$) $A=30^0$, $a=200$ feet, $c=250$ feet,
($\gamma$) $A=30^0$, $a=200$ feet, $c=125$ feet.

$$\log 2={\cdot}3010300,\quad L\sin 38^0\,41'=9{\cdot}7958800,$$
$$\log 6{\cdot}0389={\cdot}7809578,\quad L\sin 8^0\,41'=9{\cdot}1789001,$$
$$\log 6{\cdot}0390={\cdot}7809650.$$

266. It saves a little trouble in practice when using the formula $a = \frac{b \sin A}{\sin B}$, to write it thus; $a = b \sin A \,.\, \operatorname{cosec} B$.

For then, $\log a = \log b + L \sin A + L \operatorname{cosec} B - 20$.

Thus the *subtraction* of a logarithm is avoided.

*267. In the following Examples the student must find the necessary logarithms etc. from the Tables.

## *MISCELLANEOUS EXAMPLES. LXX.

(1) Find $A$ when $a = 374{\cdot}5$, $b = 576{\cdot}2$, $c = 759{\cdot}3$ feet.

(2) Find $B$ when $a = 4001$, $b = 9760$, $c = 7942$ yards.

(3) Find $C$ when $a = 8761{\cdot}2$, $b = 7643$, $c = 4693{\cdot}8$ chains.

(4) Find $B$ when $A = 86^0\ 19'$, $b = 4930$, $c = 5471$ chains.

(5) Find $C$ when $B = 32^0\ 58'$, $c = 1873{\cdot}5$, $a = 764{\cdot}2$ chains.

(6) Find $c$ when $C = 108^0\ 27'$, $a = 36541$, $b = 89170$ feet.

(7) Find $c$ when $B = 74^0\ 10'$, $C = 62^0\ 45'$, $b = 3720$ yards.

(8) Find $b$ when $B = 100^0\ 19'$, $C = 44^0\ 59'$, $a = 1000$ chains.

(9) Find $a$ when $B = 123^0\ 7'\ 20''$, $C = 15^0\ 9'$, $c = 9964$ yards.

Find the other two angles in the six following triangles.

(10) $C = 100^0\ 37'$, $b = 1450$, $c = 6374$ chains.

(11) $C = 52^0\ 10'$, $b = 643$, $c = 872$ chains.

(12) $A = 76^0\ 2'\ 30''$, $b = 1000$, $a = 2000$ chains.

(13) $C = 54^0\ 23'$, $b = 873{\cdot}4$, $c = 752{\cdot}8$ feet.

(14) $C = 18^0\ 21'$, $b = 674{\cdot}5$, $c = 269{\cdot}7$ chains.

(15) $A = 29^0\ 11'\ 43''$, $b = 7934$, $a = 4379$ feet.

(16) The difference between the angles at the base of a triangle is $17^0\ 48'$, and the sides subtending those angles are $105{\cdot}25$ feet and $76{\cdot}75$ feet; find the third angle.

(17) If $b : c = 4 : 5$, $a = 1000$ yards and $A = 37^0\ 19'$, find $b$.

The student will find some Examples of Solution of Triangles without the aid of logarithms, in an Appendix.

## *MISCELLANEOUS EXAMPLES. LXX b.

In the following Examples Logarithms are not required; they are not *practical* Examples, but some of them require for their solution considerable neatness and ingenuity; for which reason they are often set in examinations.

### PART I.

(1) Simplify the formulæ

$$\cos A = \frac{b^2+c^2-a^2}{2bc}, \qquad \cos \tfrac{1}{2}A = \sqrt{\left\{\frac{s(s-a)}{bc}\right\}}$$

in the case of an equilateral triangle.

(2) The sides of a triangle are as $2 : \sqrt{6} : 1+\sqrt{3}$, find the angles.

(3) The sides of a triangle are as 4, $2\sqrt{2}$, $2(\sqrt{3}-1)$, find the angles.

(4) Given $C=120^0$, $c=\sqrt{19}$, $a=2$, find $b$.

(5) Given $A=60^0$, $b=4\sqrt{7}$, $c=6\sqrt{7}$, find $a$.

(6) Given $A=45^0$, $B=60^0$ and $a=2$, find $c$.

(7) The sides of a triangle are as $7 : 8 : 13$, find the greatest angle.

(8) The sides of a triangle are 1, 2, $\sqrt{7}$, find the greatest angle.

(9) The sides of a triangle are as $a : b : \sqrt{(a^2+ab+b^2)}$, find the greatest angle.

(10) When $a : b : c$ as $3 : 4 : 5$, find the greatest and least angles; given $\cos 36^0\ 52' = \cdot 8$.

(11) If $a=5$ miles, $b=6$ miles, $c=10$ miles, find the greatest angle. [$\cos 49^0\ 33' = \cdot 65$.]

(12) If $a=4$, $b=5$, $c=8$, find $C$; given that $\cos 54^0 54' = \cdot 575$.

(13) The two sides $a$ and $b$ are $\sqrt{3}+1$ and 2 respectively; $C=30^0$; solve the triangle.

(14) Given $C=18^0$, $a=\sqrt{5}+1$, $c=\sqrt{5}-1$, find the other angles.

(15) If $b=3$, $C=120^0$, $c=\sqrt{13}$, find $a$ and the sines of the other angles.

(16) Given $A=105^0$, $B=45^0$, $c=\sqrt{2}$, solve the triangle.

(17) Given $B=75^0$, $C=30^0$, $c=\sqrt{8}$, solve the triangle.

(18) Given $B=45^0$, $c=\sqrt{75}$, $b=\sqrt{50}$, solve the triangle.

(19) Given $B=30^0$, $c=150$, $b=50\sqrt{3}$, show that of the two triangles which satisfy the data one will be isosceles and the other right-angled. Find the third side in the greatest of these triangles.

(20) Is the solution ambiguous when $B=30^0$, $c=150$, $b=75$?

(21) If the angles adjacent to the base of a triangle are $22\frac{1}{2}^0$ and $112\frac{1}{2}^0$, show that the perpendicular altitude will be half the base.

(22) If $a=2$, $b=4-2\sqrt{3}$, $c=\sqrt{6}(\sqrt{3}-1)$, solve the triangle.

(23) If $A=9^0$, $B=45^0$, $b=\sqrt{6}$, find $c$.

(24) Given $B=15^0$, $b=\sqrt{3}-1$, $c=\sqrt{3}+1$, solve the triangle.

(25) Given $\sin B = \cdot 25$, $a=5$, $b=2\cdot 5$, find $A$. Draw a figure to explain the result.

(26) Given $C=15^0$, $c=4$, $a=4+\sqrt{48}$, solve the triangle.

(27) Two sides of a triangle are $3\sqrt{6}$ yards and $3(\sqrt{3}+1)$ yards, and the included angle $45^0$, solve the triangle.

(28) If $C=30^0$, $b=100$, $c=45$, is the triangle ambiguous?

(29) Prove that if $A=45^0$ and $B=60^0$ then $2c=a(1+\sqrt{3})$.

(30) The cosines of two of the angles of a triangle are $\frac{1}{4}$ and $\frac{3}{5}$, find the ratio of the sides.

# CHAPTER XVIII.

## ON THE MEASUREMENT OF HEIGHTS AND DISTANCES.

268. We have said (Art. 97) that the measurement, with scientific accuracy, of a line of any considerable length involves a long and difficult process.

On the other hand, sometimes it is required to find the *direction* of a line that it may point to an object which is not visible from the point from which the line is drawn. As for example when a tunnel has to be constructed.

By the aid of the Solution of Triangles

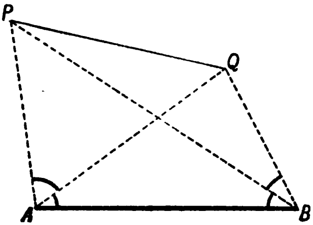
we can find the length of the distance between points which are inaccessible;

we can calculate the magnitude of angles which cannot be practically observed;

we can find the relative heights of distant and inaccessible points.

The method on which the Trigonometrical Survey of a country is conducted affords the following illustration.

269. *To find the distance between two distant objects.*



Two convenient positions $A$ and $B$, on a level plain as far apart as possible, having been selected, the distance between $A$ and $B$ is measured with the greatest possible care. This line $AB$ is called the **base line.** (In the survey of England, the base line is on Salisbury plain, and is about 36,578 feet long).

Next, the two distant objects, $P$ and $Q$ (church spires, for instance) visible from $A$ and $B$, are chosen.

The angles $PAB$, $PBA$ are observed. Then by Case II. Chapter XVII, the lengths of the lines $PA$, $PB$ are calculated.

Again, the angles $QAB$, $QBA$ are observed; and by Case II. the lengths of $QA$ and $QB$ are calculated.

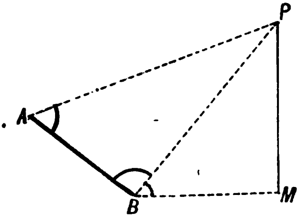
Thus the lengths of $PA$ and $QA$ are found.

The angle $PAQ$ is observed; and then by Case III. the length of $PQ$ is calculated.

270. Thus the distance between two points $P$ and $Q$ has been found. The points $P$ and $Q$ are not necessarily accessible; the only condition being that $P$ and $Q$ must be *visible* from both $A$ and $B$.

271. In practice, the points $P$ and $Q$ will generally be accessible, and then the line $PQ$, whose length has been calculated, may be used as a new base to find other distances.

272. *To find the height of a distant object above the point of observation.*



Let $B$ be the point of observation; $P$ the distant object. From $B$ measure a base line $BA$ of any convenient length, in any convenient direction; observe the angles $PAB$, $PBA$, and by Case II. calculate the length of $BP$. Next observe at $B$ the 'angle of elevation' of $P$; that is, the angle which the line $BP$ makes with the horizontal line $BM$, $M$ being the point in which the vertical line through $P$ cuts the horizontal plane through $B$.

Then $PM$, which is the vertical height of $P$ above $B$ can be calculated, for $PM = BP \, . \sin MBP$.

*Example* 1. *The distance between a church spire* A *and a milestone* B *is known to be* 1764·3 *feet; C is a distant spire. The angle* CAB *is* $94^0\ 54'$. *and the angle* CBA *is* $66^0\ 39'$. *Find the distance of* C *from* A.

$ABC$ is a triangle and we know one side $c$ and two angles ($A$ and $B$), and therefore it can be solved by Case II.

$$\text{The angle } ACB = 180^0 - 94^0\ 54' - 66^0\ 39'$$
$$= 18^0\ 27'.$$

Therefore the triangle is the same as that solved on page 210. Therefore $AC = 5118{\cdot}2$ feet.

*Example 2. If the spire* C *in the last Example stands on a hill, and the angle of elevation of its highest point is observed at* A *to be* $4^{\circ}\ 19'$*; find how much higher* C *is than* A.

The required height $x = AC \,.\, \sin 4^{\circ}\ 19'$ [Art. 268], and $AC$ is 5118·2 feet,

$$\begin{aligned} \therefore\ \log x &= \log (AC \,.\, \sin 4^{\circ}\ 19') \\ &= \log 5118{\cdot}2 + L \sin 4^{\circ}\ 19' - 10 \\ &= 3{\cdot}7091173 + 8{\cdot}8766150 - 10 \\ &= 2{\cdot}5857323 = \log 385{\cdot}24. \end{aligned}$$

Therefore $x = 385$ ft. 3 in. nearly.

## EXAMPLES. LXXI.

(Exercises XIV. and LXI. consist of easy examples on this subject).

(1) Two straight roads inclined to one another at an angle of $60^{\circ}$, lead from a town $A$ to two villages $B$ and $C$; $B$ on one road distant 30 miles from $A$, and $C$ on the other road distant 15 miles from $A$. Find the distance from $B$ to $C$. Ans. 25·98 m.

(2) Two ships leave harbour together, one sailing N.E. at the rate of $7\frac{1}{2}$ miles an hour and the other sailing North at the rate of 10 miles an hour. Prove that the distance between the ships after an hour and a half is 10·6 miles.

(3) $A$ and $B$ are two consecutive milestones on a straight road and $C$ is a distant spire. The angles $ABC$ and $BAC$ are observed to be $120^{\circ}$ and $45^{\circ}$ respectively. Show that the distance of the spire from $A$ is 3·346 miles.

(4) If the spire $C$ in the last question stands on a hill, and its angle of elevation at $A$ is $15^{\circ}$, show that it is ·866 of a mile higher than $A$.

(5) If in Question (3) there is another spire $D$ such that the angles $DBA$ and $DAB$ are $45^{\circ}$ and $90^{\circ}$ respectively and the angle $DAC$ is $45^{\circ}$; prove that the distance from $C$ to $D$ is $2\frac{3}{4}$ miles very nearly.

(6) $A$ and $B$ are two consecutive milestones on a straight road, and $C$ is the chimney of a house visible from both $A$ and $B$. The angles $CAB$ and $CBA$ are observed to be $36^\circ\, 18'$ and $120^\circ\, 27'$ respectively. Show that $C$ is 2639·5 yards from $B$,

$\log 1760 \;\;= 3{\cdot}2455127 \qquad L \sin 36^\circ\, 18' = \;\; 9{\cdot}7723314$

$\log 2639{\cdot}5 = 3{\cdot}42152 \qquad L \operatorname{cosec} 23^\circ\, 15' = 10{\cdot}4036846.$

(7) $A$ and $B$ are two points on opposite sides of a mountain, and $C$ is a place visible from both $A$ and $B$. It is ascertained that $C$ is distant 1794 feet and 3140 feet from $A$ and $B$ respectively and the angle $ACB$ is $58^\circ\, 17'$. Show that the angle which the line pointing from $A$ to $B$ makes with $AC$ is $86^\circ\, 55'\, 49''$,

$\log 1346 = 3{\cdot}1290451 \qquad L \cot 29^\circ\, 8'\, 30'' = 10{\cdot}2537194$

$\log 4934 = 3{\cdot}6931991 \qquad L \tan 26^\circ\, 4'\, 19'' = \;\; 9{\cdot}6895654$

(8) $A$ and $B$ are two hill-tops 34920 feet apart, and $C$ is the top of a distant hill. The angles $CAB$ and $CBA$ are observed to be $61^\circ\, 53'$ and $76^\circ\, 49'$ respectively. Prove that the distance from $A$ to $C$ is 51515 feet,

$\log 34920 = 4{\cdot}5430742 \qquad L \sin 76^\circ\, 49' = \;\; 9{\cdot}9884008$

$\log 51515 = 4{\cdot}71193 \qquad L \operatorname{cosec} 41^\circ\, 18' = 10{\cdot}1804552.$

(9) From two stations $A$ and $B$ on shore, 3742 yards apart, a ship $C$ is observed at sea. The angles $BAC$, $ABC$ are simultaneously observed to be $72^\circ\, 34'$ and $81^\circ\, 41'$ respectively. Prove that the distance from $A$ to the ship is 8522·7 yards,

$\log 3742 = 3{\cdot}5731038 \qquad L \sin 81^\circ\, 41' = \;\; 9{\cdot}9954087$

$\log 8522{\cdot}7 = 3{\cdot}9305774 \qquad L \operatorname{cosec} 25^\circ\, 45' = 10{\cdot}3620649.$

(10) The distance between two mountain peaks is known to be 4970 yards, and the angle of elevation of one of them when seen from the other is $9^\circ\, 14'$. How much higher is the first than the second? $\sin 9^\circ\, 14' = {\cdot}1604555$. *Ans.* 797·5 yards.

(11) Two straight railways intersect at an angle of $60^\circ$. From their point of intersection two trains start, one on each line, one at the rate of 40 miles an hour. Find the rate of the second train that at the end of an hour they may be 35 miles apart. *Ans.* Either 25 or 15 miles an hour. (Art. 264.)

✓ (12) $A$ and $B$ are two positions on opposite sides of a mountain; $C$ is a point visible from $A$ and $B$; $AC$ and $BC$ are 10 miles and 8 miles respectively, and the angle $BCA$ is $60^0$. Prove that the distance between $A$ and $B$ is 9·165 miles.

✓ (13) In the last question, if the angle of elevation of $C$ at $A$ is $8^0$, and at $B$ is $2^0\ 48'\ 24''$: show that the height of $B$ above $A$ is one mile very nearly.

$$\sin 8^0 = ·1391731 \quad \sin 2^0\ 48'\ 24'' = ·0489664.$$

(14) Show that the angles which a tunnel going through the mountain from $A$ to $B$, in Questions (12) and (13), would make (i) with the horizon, (ii) with the line joining $A$ and $C$, are respectively $6^0\ 16'$ and $49^0\ 6'\ 24''$.

$$\sin 6^0\ 16' = ·1091\ ; \quad \tan 10^0\ 53'\ 36'' = ·192450.$$

(15) $A$ and $B$ are consecutive milestones on a straight road; $C$ is the top of a distant mountain. At $A$ the angle $CAB$ is observed to be $38^0\ 19'$; at $B$ the angle $CBA$ is observed to be $132^0\ 42'$, and the angle of elevation of $C$ at $B$ is $10^0\ 15'$. Show that the top of the mountain is 1243·5 yards higher than $B$.

$L \sin 38^0\ 19' = 9·7923968$ $\quad \log 1760 = 3·2455127$

$L \operatorname{cosec} 8^0\ 59' = 10·8064659$ $\quad \log 1243·5 = 3·09465$

$L \sin 10^0\ 15' = 9·2502822.$

(16) A base line $AB$, 1000 feet long is measured along the straight bank of a river; $C$ is an object on the opposite bank; the angles $BAC$, and $CBA$ are observed to be $65^0\ 37'$ and $53^0\ 4'$ respectively. Prove that the perpendicular breadth of the river at $C$ is 829·87 feet; having given

$L \sin 65^0\ 37' = 9·9594248,$ $\quad L \sin 53^0\ 4' = 9·9027289$

$L \operatorname{cosec} 61^0\ 19' = 10·0568589,$ $\quad \log 8·2987 = ·91901.$

## * MISCELLANEOUS EXAMPLES. LXXII.

(1) A man walking along a straight road at the rate of three miles an hour sees, in front of him at an elevation of $60^0$, a balloon which is travelling horizontally in the same direction at the rate of six miles an hour; ten minutes after he observes that the elevation is $30^0$. Prove that the height of the balloon above the road is $440\sqrt{3}$ yards.

(2) A person standing at a point $A$, due south of a tower built on a horizontal plain, observes the altitude of the tower to be $60^0$. He then walks to a point $B$ due west from $A$ and observes the altitude to be $45^0$, and then at the point $C$ in $AB$ produced he observes the altitude to be $30^0$. Prove that $AB = BC$.

(3) The angle of elevation of a balloon, which is ascending uniformly and vertically, when it is one mile high is observed to be $35^0\,20'$; 20 minutes later the elevation is observed to be $55^0\,40'$. How fast is the balloon moving?

*Ans.* $3(\sin 20^0\,20')(\sec 55^0\,40')(\operatorname{cosec} 35^0\,20')$ miles per hour.

(4) The angular elevation of a tower at a place $A$ due south of it is $30^0$; and at a place $B$ due west of $A$, and at a distance $a$ from it, the elevation is $18^0$; show that the height of the tower is

$$\frac{a}{\sqrt{2(1+\sqrt{5})}}.$$

(5) The angular elevation of the top of a steeple at a place due south of it is $45^0$, and at another place due west of the former station and distant $a$ feet from it the elevation is $15^0$; show that the height of the steeple is $\frac{a}{2}(3^{\frac{1}{4}} - 3^{-\frac{1}{4}})$ feet.

(6) A tower stands at the foot of an inclined plane whose inclination to the horizon is $9^0$; a line is measured up the incline from the foot of the tower of 100 feet in length. At the upper extremity of this line the tower subtends an angle of $54^0$. Find the height of the tower. *Ans.* 114·4 ft.

(7) The altitude of a certain rock is observed to be $47^0$, and after walking 1000 feet towards the rock, up a slope inclined at an angle of $32^0$ to the horizon, the observer finds that the altitude is $77^0$. Prove that the vertical height of the rock above the first point of observation is 1034 ft. $\sin 47^0 = \cdot 73135$.

(8) At the top of a chimney 150 feet high standing at one corner of a triangular yard, the angle subtended by the adjacent sides of the yard are $30^0$ and $45^0$ respectively; while that subtended by the opposite side is $30^0$. Show that the lengths of the sides are 150 ft. 86·6 ft. and 106 ft. respectively.

(9) A flag staff $h$ feet high stands on the top of a tower. From a point in the plain on which the tower stands the angles of elevation of the top and bottom of the flagstaff are observed to be $a$ and $\beta$ respectively. Prove that the height of the tower is $\dfrac{h \tan \beta}{\tan a - \tan \beta}$ feet, i.e. $\dfrac{h \sin \beta . \cos a}{\sin (a - \beta)}$ feet.

(10) From the top of a cliff $h$ feet high the angles of depression of two ships at sea in a line with the foot of the cliff are $a$ and $\beta$ respectively. Show that the distance between the ships is $h (\cot \beta - \cot a)$ feet.

(11) The angular elevation of a tower at a place due south of it is $a$, and at another place due west of the first and distant $d$ from it, the elevation is $\beta$. Prove that the height of the tower is

$$\frac{d}{\sqrt{\cot^2 \beta - \cot^2 a}}, \text{ i.e. } \frac{d \sin a . \sin \beta}{\sqrt{\sin (a - \beta) . \sin (a + \beta)}}.$$

(12) A man stands on the top of the wall of height $h$, and observes the angular elevation ($a$) of the top of a telegraph post; he then descends from the wall, and finds that the angular elevation is now $\beta$; prove that the height of the post exceeds the height of the man by $h \dfrac{\sin \beta . \cos a}{\sin (\beta - a)}$.

(13) Find the height of a cloud by observing (i) its elevation ($\alpha$) and (ii) the depression ($\beta$) of its reflexion in a lake $h$ feet below the point of observation. *Ans.* Height above the lake $h \sin(\alpha+\beta) \operatorname{cosec}(\beta-\alpha)$ feet.

(14) If the angular elevation of the summits of two spires (which appear in a straight line) is $\alpha$, and the angular depressions of their reflexions in a lake, $h$ feet below the point of observation, are $\beta$ and $\gamma$, then the horizontal distance between the spires is $2h \cos^2 \alpha \sin(\beta-\gamma) . \operatorname{cosec}(\beta-\alpha) . \operatorname{cosec}(\gamma-\alpha)$ feet.

(15) A statue $AB$ on the top of a pillar $BC$ situated on level ground is found to subtend the greatest angle ($\alpha$) at the eye of an observer $E$ when he is distant $c$ feet from the pillar. Prove that the height of the statue is ($2c \tan \alpha$) feet; and, if $b$ be the height of the observer's eye from the ground find the height of the pillar. (See Note in the Answers.)

(16) A man walking along a straight road observes that the greatest angle which two objects subtend at his eye is $\alpha$. From the place where this happens he walks $a$ yards; the objects then appear in a straight line making a right angle with the road. Prove that the distance between the objects is ($2a \tan \alpha$) yards.

(17) A man standing on a horizontal plain at a distance $h$ feet from a tower, observes that a flagstaff on the tower subtends an angle $\alpha$ at his eye, and that on walking $2k$ feet towards the tower the flagstaff subtends again the same angle. Prove that the height of the flagstaff is $2(h-k) \tan \alpha$ feet.

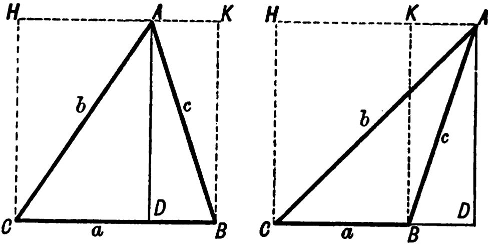
(18) A man walking along a straight road observes that the greatest angle which two objects subtend at his eye is $\alpha$; from the place where this happens he walks $a$ yards, and the objects then appear in a straight line making an angle $\beta$ with the road. Prove that the distance between the objects is $\dfrac{2a \sin\alpha . \sin\beta}{\cos\alpha+\cos\beta}$ yds.

# CHAPTER XIX.

## ON TRIANGLES AND CIRCLES.

273. *To find the* **Area** *of a Triangle.*

The area of the triangle $ABC$ is denoted by $\Delta$.



Through $A$ draw $HK$ parallel to $BC$, and through $ABC$ draw lines $AD$, $BK$, $CH$ perpendicular to $BC$.

The area of the triangle $ABC$ is half that of the rectangular parallelogram $BCHK$ [Euc. I. 41].

Therefore
$$\Delta = \frac{BC \, . \, CH}{2} = \frac{BC \, . \, DA}{2}$$

$$= \frac{a \, . \, b \sin C}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{i}).$$

But
$$\sin C = \frac{2}{ab} \, . \, \sqrt{s(s-a)(s-b)(s-c)}\,;$$

$$\therefore \; \Delta = \sqrt{s(s-a)(s-b)(s-c)} = S \ldots\ldots (\text{ii}).$$

### 274. *To find the Radius of the* **Circumscribing Circle.**

Fig. I.

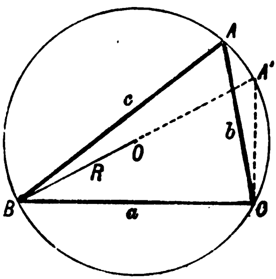
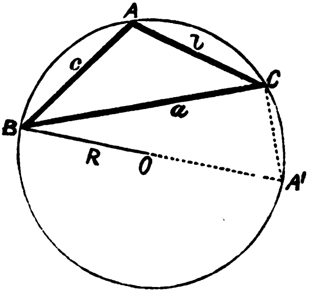


Fig. II.



Let a circle $AA'CB$ be described about the triangle $ABC$. Let $R$ stand for its radius. Let $O$ be its centre. Join $BO$, and produce it to cut the circumference in $A'$. Join $A'C$.

Then, Fig. I. the angles $BAC$, $BA'C$ in the same segment are equal; Fig. II. the angles $BAC$, $BA'C$ are supplementary; also the angle $BCA'$ in a semicircle is a right angle.

Therefore $\dfrac{CB}{A'B} = \sin CA'B = \sin CAB = \sin A$,

or,
$$\frac{a}{2R} = \sin A\,;$$

$$\therefore\ 2R = \frac{a}{\sin A}.$$

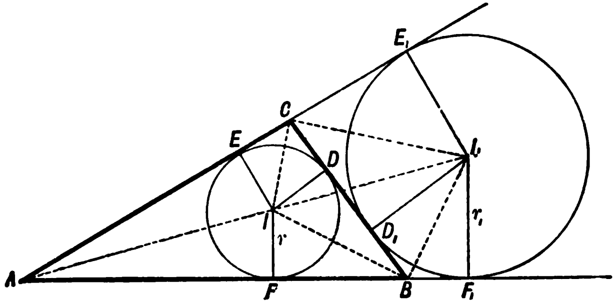
275. Similarly, it may be proved that

$$2R = \frac{b}{\sin B}\,;\ \text{and that } 2R = \frac{c}{\sin C}.$$

Hence,
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Thus $d$, the value of each of these fractions, is the diameter of the circumscribing circle.

### 276. *To find the radius of the* **Inscribed Circle.**



Let $D$, $E$, $F$ be the points in which the circle inscribed in the triangle $ABC$ touches the sides. Let $I$ be the centre of the circle; let $r$ be its radius. Then $ID = IE = IF = r$.

The area of the triangle $ABC$

$$= \text{area of } IBC + \text{area of } ICA + \text{area of } IAB.$$

And the area of the triangle $IBC = \frac{1}{2}ID \,.\, BC = \frac{1}{2}r \,.\, a$,

$$\therefore \text{ area of } ABC = \tfrac{1}{2}ID \,.\, BC + \tfrac{1}{2}IE \,.\, CA + \tfrac{1}{2}IF \,.\, AB$$

$$= \tfrac{1}{2}\,ra + \tfrac{1}{2}\,rb + \tfrac{1}{2}\,rc;$$

or,
$$\Delta = \tfrac{1}{2}r\,(a + b + c) = \tfrac{1}{2}r \,.\, 2s = rs.$$

$$\therefore r = \frac{\Delta}{s} = \frac{S}{s}.$$

277. A circle which touches one of the sides of a triangle and the other two sides produced is called an **Escribed Circle** of the triangle.

278. *To find the radius of an* **Escribed Circle.**

Let an escribed circle touch the side $BC$ and the sides $AC$, $AB$ produced in the points $D_1$, $E_1$, $F_1$ respectively. Let $I_1$ be its centre, $r_1$ its radius. Then

$$I_1D_1 = I_1E_1 = I_1F_1 = r_1.$$

The area of the triangle $ABC$

$$= \text{area of } ABI_1C - \text{area of } I_1BC,$$

$$= \text{area of } I_1CA + \text{area of } I_1AB - \text{area of } I_1BC,$$

or
$$\Delta = \tfrac{1}{2}I_1E_1 \,.\, CA + \tfrac{1}{2}I_1F_1 \,.\, AB - \tfrac{1}{2}I_1D_1 \,.\, BC$$

$$= \tfrac{1}{2}r_1b + \tfrac{1}{2}r_1c - \tfrac{1}{2}r_1a$$

$$= \tfrac{1}{2}r_1(b + c - a) = \tfrac{1}{2}r_1(2s - 2a)$$

$$= r_1(s - a).$$

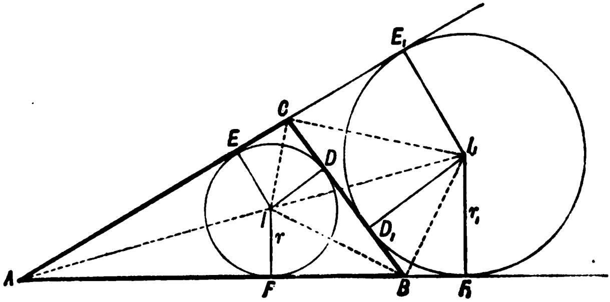
$$\therefore\ r_1 = \frac{\Delta}{s-a} = \frac{S}{s-a}.$$

279. Similarly if $r_2$ and $r_3$ be the radii of the other two escribed circles of the triangle $ABC$, then

$$r_2 = \frac{S}{s-b}; \quad r_3 = \frac{S}{s-c}.$$

*280. The following results are often useful.

With the construction of the last two articles; the lines $IA$, $IB$, $IC$ bisect the angles $A$, $B$, $C$ respectively; the lines $I_1A$, $I_1B$, $I_1C$ bisect the angles $BAC$, $CBF_1$, $BCE_1$ respectively; so that $AII_1$ is a straight line.



(i) Two tangents drawn from an external point to touch a circle are equal; therefore

$$AE = AF;\quad BD = BF;\quad CE = CD;$$

$$AE_1 = AF_1;\ BD_1 = BF_1;\ CE_1 = CD_1.$$

(ii) $2AE_1 = AE_1 + AF_1 = (AC + CD_1) + (AB + BD_1)$

$$= AC + CB + BA = a + b + c = 2s.$$

(iii) $2AE = AE + AF = (AC - CD) + (AB - BD)$

$$= AC + AB - BC = b + c - a = 2(s - a).$$

Similarly, $BF = BD = (s - b)$; $CD = CE = (s - c)$.

(iv) $BD_1 = BF_1 = AF_1 - AB = (s - c)$.

Similarly, $CE_1 = CD_1 = (s - b)$.

(v) Hence, $BD_1 = CD$, and $CD_1 = BD$.

(vi) $FF_1 = AF_1 - AF = s - (s - a) = a$.

Similarly, $EE_1 = a$.

(vii) $II_1 = AI_1 - AI = AF_1 \sec\frac{A}{2} - AF\sec\frac{A}{2} = FF_1 \sec\frac{A}{2}$

$$\therefore\ II_1 = \frac{a}{\cos\frac{A}{2}}.$$

(viii) $IF = AF \tan\frac{A}{2}$; or, $r = (s-a)\tan\frac{A}{2}$,

$I_1F = AF_1 \tan\frac{A}{2}$; or, $r_1 = s\tan\frac{A}{2}$.

(ix) $AF = IF\cot\frac{A}{2}$, and $BF = IF\cot\frac{B}{2}$;

$$\therefore\ c = AF + BF = r\left(\cot\frac{A}{2} + \cot\frac{B}{2}\right);$$

$$\therefore\ r = \frac{c}{\cot\frac{A}{2} + \cot\frac{B}{2}} = \frac{c\sin\frac{A}{2}.\sin\frac{B}{2}}{\sin\frac{A+B}{2}} = \frac{c\sin\frac{A}{2}.\sin\frac{B}{2}}{\cos\frac{C}{2}}.$$

(x) $F_1BC = 180° - B$, $\therefore\ I_1BC = 90° - \frac{B}{2}$. Hence,

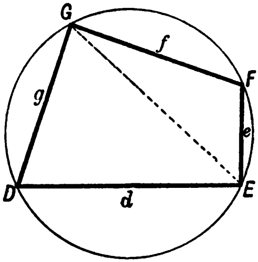
$$r_1\left\{\cot\left(90° - \frac{B}{2}\right) + \cot\left(90° - \frac{C}{2}\right)\right\} = a;$$

or,
$$r_1 = \frac{a\cos\frac{B}{2}.\cos\frac{C}{2}}{\cos\frac{A}{2}}.$$

(xi) $r_1 = AF_1\tan\frac{A}{2}$, and $r = AF\tan\frac{A}{2}$; therefore

$$r_1 - r = FF_1\tan\frac{A}{2} = a\tan\frac{A}{2}.$$

*281. *To find an expression for the* **area** *of a quadrilateral inscribed in a circle, in terms of the sides.*



Let $DEFG$ be the quadrilateral; and let $d, e, f, g$ represent the sides.

The angle $F = 180° - D$. [Euc. III. 22.]

The area of $GDE = \frac{1}{2}dg \sin D$. [Art. 273.]

The area of $EFG = \frac{1}{2}ef \sin F = \frac{1}{2}ef \sin(180° - D)$

$$= \frac{1}{2}ef \sin D.$$

Thus the area of the quadrilateral is

$$\tfrac{1}{2}(dg + ef) \sin D.$$

We must find an expression for $\sin D$ in terms of the sides.

From the triangle $EGD$ we have

$$EG^2 = d^2 + g^2 - 2dg \cos D. \qquad \text{[Art. 239.]}$$

From the triangle $EGF$ we have

$$EG^2 = e^2 + f^2 - 2ef \cos F$$

$$= e^2 + f^2 + 2ef \cos D; \qquad [\text{For } F = 180° - D]$$

$$\therefore\ d^2 + g^2 - 2dg \cos D = e^2 + f^2 + 2ef \cos D;$$

$$\therefore\ 2\,(dg+ef)\cos D = d^2+g^2-e^2-f^2\,;$$

$$\therefore\ 1-\cos^2 D = 1-\frac{(d^2+g^2-e^2-f^2)^2}{4\,(dg+ef)^2}\,;$$

$$\therefore\ \sin^2 D = \frac{4\,(dg+ef)^2-(d^2+g^2-e^2-f^2)^2}{4\,(dg+ef)^2}\,.$$

The numerator of this expression is equal to

$$\{(2dg+2ef)+(d^2+g^2-e^2-f^2)\}\,\{(2dg+2ef)-(d^2+g^2-e^2-f^2)\}$$

$$=\{(d+g)^2-(e-f)^2\}\times\{(e+f)^2-(d-g)^2\}$$

$$=\{(d+g)+(e-f)\}\,\{(d+g)-(e-f)\}$$
$$\times\{(e+f)+(d-g)\}\,\{(e+f)-(d-g)\}$$

$$=(d+g+e-f)\,(d+g-e+f)\,(e+f+d-g)\,(e+f+g-d).$$

Let 2s stand for $(d+e+f+g)$.

Then the above expression may be written

$$(2s-2f)\,(2s-2e)\,(2s-2g)\,(2s-2d).$$

Therefore $\sin^2 D = \dfrac{4\,(s-d)\,(s-e)\,(s-f)\,(s-g)}{(dg+ef)^2}$.

or $\sin D = \dfrac{2\sqrt{(s-d)\,(s-e)\,(s-f)\,(s-g)}}{dg+ef}$.

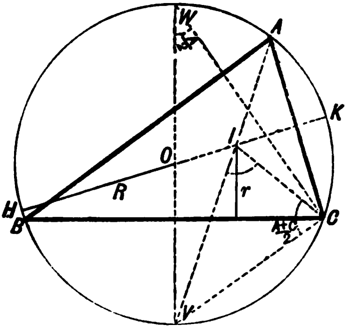
Hence the area of the quadrilateral *DEFG*

which is $\frac{1}{2}\,(dg+ef)\sin D$

$$=\sqrt{(s-d)\,(s-e)\,(s-f)\,(s-g)}.$$

* 282. *To prove that if* I *and* O *are the centres of the inscribed and circumscribed circles of a triangle, then*

$$OI^2 = R^2 - 2Rr.$$



Let $OI$ produced cut the circumscribing circle in $H$ and $K$. Join $IA$, $IC$; produce $IA$ to cut the circle $ABC$ again in $V$. Draw the diameter $VOW$ and join $CW$. Then the angle $VIC = IAC + ICA = \frac{1}{2}A + \frac{1}{2}C = \frac{1}{2}(A + C)$. The angle

$$ICV = ICB + BCV = ICB + BAV = \tfrac{1}{2}C + \tfrac{1}{2}A = \tfrac{1}{2}(A + C).$$

Therefore the angle $VCI = VIC$, and $\therefore VI = VC$.

Now $OK^2 - OI^2 = HI . IK$ [Euc. II. 5]

$= VI . IA = VC . IA.$ [Euc. III. 35.]

But $VC = VW . \sin CWV = 2R . \sin CAV = 2R . \sin \dfrac{A}{2}$,

and $IA = \dfrac{r}{\sin \dfrac{A}{2}}$. [See Figure on page 234.]

Hence $OK^2 - OI^2 = 2R \sin \dfrac{A}{2} \times \dfrac{r}{\sin \dfrac{A}{2}}$,

or $R^2 - OI^2 = 2Rr.$

## EXAMPLES. LXXIII.

(1) Find the area of the triangle $ABC$ when

(i) $a=4$, $b=10$ feet, $C=30^0$.

(ii) $b=5$, $c=20$ inches, $A=60^0$.

(iii) $c=66\frac{2}{3}$, $a=15$ yards, $B=17^0\ 14'$ [$\sin 17^0\ 14'=\cdot 29626$].

(iv) $a=13$, $b=14$, $c=15$ chains.

(v) $a=10$, the perpendicular from $A$ on $BC=20$ feet.

(vi) $a=625$, $b=505$, $c=904$ yards.

(2) Find the Radii of the Inscribed and each of the Escribed Circles of the triangle $ABC$ when $a=13$, $b=14$, $c=15$ feet.

(3) Show that the triangles in which (i) $a=2$, $A=60^0$; (ii) $b=\frac{2}{3}.\sqrt{3}$, $B=30^0$ can be inscribed in the same circle.

(4) Prove that $R=\dfrac{abc}{4S}$; find $R$ in the triangle of (2).

(5) Prove that if a series of triangles of equal perimeter are described about the same circle, they are equal in area.

(6) If $A=60^0$, $a=\sqrt{3}$, $b=\sqrt{2}$, prove that the area

$$=\tfrac{1}{4}(3+\sqrt{3}).$$

(7) Prove that each of the following expressions represents the area of the triangle $ABC$:

(i) $\dfrac{abc}{4R}$. (ii) $2R^2 \sin A . \sin B . \sin C$.

(iii) $rs$. (iv) $Rr(\sin A+\sin B+\sin C)$.

(v) $\frac{1}{2}a^2 \sin B . \sin C . \operatorname{cosec} A$. (vi) $ra \operatorname{cosec}\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$.

(vii) $(rr_1r_2r_3)^{\frac{1}{2}}$. (viii) $\frac{1}{2}(a^2-b^2)\sin A . \sin B . \operatorname{cosec}(A-B)$.

Prove the following statements:

(8) If $a$, $b$, $c$ are in A. P., then $ac=6rR$.

(9) The area of the greatest triangle, two of whose sides are 50 and 60 feet, is 1500 sq. feet.

(10) If the altitude of an isosceles triangle is equal to the base, $R$ is five-eighths of the base.

(11) $R(\sin A+\sin B+\sin C)=s$.

(12) $bc=4R^2(\cos A+\cos B.\cos C)$.

(13) If $C$ is a right angle, $2r+2R=a+b$.

(14) $r_2r_3+r_3r_1+r_1r_2=s^2$.

(15) $\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}=\frac{1}{2rR}$.

(16) $r_1\cot\frac{A}{2}=r_2\cot\frac{B}{2}=r_3\cot\frac{C}{2}=r\cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2}$.

(17) $r_1+r_2=c\cot\frac{C}{2}$.

(18) $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}$.

(19) $\frac{r_1-r}{a}+\frac{r_2-r}{b}=\frac{c}{r_3}$. (20) $r_1+r_2+r_3-r=4R$.

(21) $a.b.c.r=4R(s-a)(s-b)(s-c)$.

(22) The distances of the centres of the escribed circles of the triangle $ABC$ from that of the inscribed circle are

$$4R\sin\frac{A}{2},\ 4R\sin\frac{B}{2},\ 4R\sin\frac{C}{2}.$$

(23) If $A$ is a right angle, $r_2+r_3=a$.

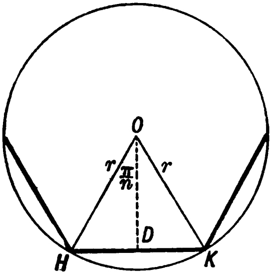
(24) In an equilateral triangle $3R=6r=2r_1$.

(25) $\frac{r_1}{bc}+\frac{r_2}{ca}+\frac{r_3}{ab}=\frac{1}{r}-\frac{1}{2R}$.

# * CHAPTER XX.

## ON THE AREA OF THE CIRCLE, THE CONSTRUCTION OF TRIGONOMETRICAL TABLES, &c.

283. Let $r$ be the radius of the circle described about a regular polygon of $n$ sides. Let $O$ be the centre; $HK$ one of the sides.



From $O$ draw $OD$ perpendicular to $HK$, bisecting both $HK$ and the angle $HOK$. Then, since the polygon has $n$ equal sides, $n$ times the angle $HOK$ make up four right angles

$$\therefore\ HOK = \frac{2\pi}{n}, \text{ and } \therefore\ DOK = \frac{\pi}{n}. \quad \text{Hence,}$$

the perimeter of the polygon, which is $n$ times $HK$,

$$= 2n\,.\,DH = 2n\,.\,OH \sin DOH$$

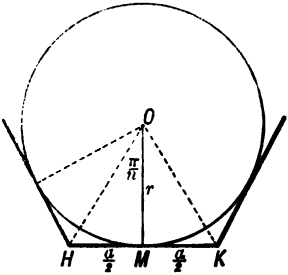
$$= 2nr \sin\frac{\pi}{n};$$

the area of the polygon, which is $n$ times the area of $HOK$,

$$= nOD\,.\,HD = nr\cos\frac{\pi}{n}\,.\,r\sin\frac{\pi}{n}$$

$$= nr^2 \sin\frac{\pi}{n}\,.\,\cos\frac{\pi}{n}.$$

284. Let $r$ be the radius of the circle inscribed in a regular polygon of $n$ sides. Let $O$ be the centre, $HK$ one of the sides.



From $O$ draw $OM$ perpendicular to $HK$, bisecting both $HK$ and the angle $HOK$. Then since the polygon has $n$ equal sides, $n$ times the angle $HOK$ make up four right angles,

$$\therefore HOK = \frac{2\pi}{n} \text{ and } \therefore MOH = \frac{\pi}{n}.$$ Hence

the perimeter of the polygon, which is $n$ times $HK$,

$$= 2n \,.\, MH = 2n \,.\, OM \tan MOH$$

$$= 2nr \tan \frac{\pi}{n}.$$

The area of the polygon, which is $n$ times the area of $HOK$,

$$= n \,.\, OM \,.\, MH = n \,.\, OM \,.\, OM \tan MOH$$

$$= nr^2 \tan \frac{\pi}{n}$$

$=$ the radius × half the perimeter.

285. Let a circle of radius $r$ have a regular polygon of $n$ sides circumscribed about it, and also a regular polygon of the same number ($n$) of sides inscribed in it.

The perimeter of the circumscribing polygon is $2nr\tan\frac{\pi}{n}$.

The perimeter of the inscribed polygon is $2nr\sin\frac{\pi}{n}$.

The ratio of these perimeters is $1 : \cos\frac{\pi}{n}$ ............(1).

The area of the circumscribed polygon is $nr^2\tan\frac{\pi}{n}$.

The area of the inscribed polygon is $nr^2\sin\frac{\pi}{n}\cos\frac{\pi}{n}$.

The ratio of these areas is $1 : \cos^2\frac{\pi}{n}$ ................ (ii).

286. We must assume the two following axioms:

(i) The circumference of the circle lies in magnitude **between** the perimeter of a circumscribed and that of an inscribed polygon.

(ii) The area of the circle lies in magnitude **between** the area of a circumscribed and that of an inscribed polygon.

Now, when $n$ is increased, $\frac{\pi}{n}$ is diminished, and therefore (by Art. 94) $\cos\frac{\pi}{n}$ approaches 1.

Hence, as the number of the sides of the two polygons in Art. 285 is increased, their perimeters approach to equality.

And since the circumference of the circle always lies in magnitude **between** them, each of these perimeters must approach the circumference of the circle.

Therefore, the **circumference** of a circle is that, to which the perimeter of a regular inscribed (or circumscribed) polygon approaches as the number of its sides is increased, and from which it can be made to differ by less than any assignable quantity however small. (Cf. Art. 32.)

287. In like manner it follows, from (ii) Art. 285, that the area of the circle is the '*limit*' of the area of a regular inscribed (or circumscribed) polygon when the number of the sides is indefinitely increased.

Now, twice the area of a polygon circumscribed about a circle is equal to (the radius × the perimeter). [Art. 284]

This is true however great be the number of the sides. It is therefore true when the number of the sides is indefinitely increased.

Therefore it is true for the circle itself. Hence,

twice the area of a circle = the radius × the circumference.

Or, the area of a circle $= \frac{1}{2}r\,(2\pi r)$ [Art. 34]

$$= \pi r^2.$$

## EXAMPLES. LXXIV.

(1) Prove the surd expressions of Art. 37 for the ratios of the perimeters of certain regular polygons to the diameters of their circumscribing circles.

(2) Prove that the area of a regular polygon of twelve sides described about a circle whose radius is 1 foot is 3·215 sq. ft.

(3) Prove that the area of a square described about a circle is $\frac{4}{3}$ of the dodecagon inscribed in the same circle.

(4) Find the perimeter of a regular polygon of 100 sides (i) when described about a circle of 1 foot diameter, (ii) when inscribed in the same circle. *Ans.* (i) 3·14263, (ii) 3·14108 ft.

(5) An equilateral triangle and a regular hexagon have the same perimeter: show that the areas of their inscribed circles are as 4 : 9.

(6) Prove that the area of a regular polygon of $n$ sides, each of whose sides is $a$, is $\frac{na^2}{4} \cdot \cot \frac{180^0}{n}$.

(7) If the areas of a regular pentagon and decagon are equal, the ratio of their sides is $\sqrt[4]{20} : 1$.

(8) If $a$ be a side of a regular polygon and $R$ and $r$ the radii of the inscribed and described circles respectively, then $2R = a \operatorname{cosec} \frac{\pi}{n}$ and $2r = a \cot \frac{\pi}{n}$.

(9) If $R$ and $r$ be the radii of the inscribed and circumscribed circles of a regular polygon of $n$ sides, each $= a$,

$$R + r = \frac{a}{2} \cot \frac{\pi}{2n}.$$

(10) The triangle formed from one side each of a regular pentagon, hexagon and decagon inscribed in the same circle, is right-angled.

(11) Prove that the area of an irregular polygon described about a circle is equal to the product of the radius and half the perimeter of the polygon.

(12) The area of an irregular polygon of an even number of sides circumscribed about a circle is equal to the radius × the sum of every alternate side.

(13) The diameter of the dome of St Paul's is 108 feet; prove that it covers an area of 1018 sq. yds.

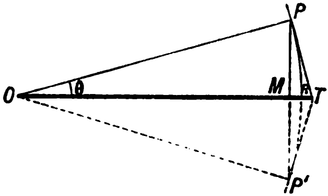
(14) The radius of the circle whose area is one acre is $39\frac{1}{4}$ yds.

(15) A length of 300 yards of paper, the thickness of which is the hundred and fiftieth part of an inch, is rolled up into a solid cylinder. Find approximately the diameter of the cylinder. *Ans.* 9·575 in.

(16) The diameter of a roll of carpet is 2 feet and the thickness of the carpet is the eighth of an inch. What is the length of the carpet? *Ans.* 301·6 ft.

## ON THE CONSTRUCTION OF TRIGONOMETRICAL TABLES.

288. *To prove that, if* $\theta$ *be the circular measure of an angle less than a right angle,* $\sin\theta$, $\theta$, $\tan\theta$ *are in ascending order of magnitude.*



Let $ROP$ be an angle ($\theta$) less than a right angle. Make the angle $ROP'$ on the other side of $OR$ equal to $ROP$. With centre $O$ and any radius $OR$ describe the arc $P'RP$. Draw $PT$, $P'T$ to touch the circle at $P$ and $P'$. $PT$ and $P'T$ will meet on $OR$. The line $PP'$ is bisected by $OR$ at right angles at $M$.

Then, since the circumference of a circle lies *between* the perimeter of the inscribed and circumscribed polygons, it follows that the arc $P'RP$ lies in magnitude between $P'MP$ and $P'T + TP$. In other words, $P'MP$, the arc $P'RP$, and $P'T + TP$ are in ascending order of magnitude.

Therefore also their halves $MP$, $RP$, $TP$ are in ascending order of magnitude.

And so also are $$\frac{MP}{OP},\ \frac{RP}{OP},\ \frac{TP}{OP}.$$

That is, $\sin\theta$, $\theta$, $\tan\theta$ are in ascending order of magnitude, where $\theta\left[=\frac{\text{arc}}{\text{radius}}\right]$ is the circular measure of the angle referred to.

289. Hence, 1, $\frac{\theta}{\sin\theta}$, $\frac{1}{\cos\theta}$ are in ascending order of magnitude.

290. *To prove that the 'limit' of* $\frac{\theta}{\sin\theta}$, *when* $\theta=0$, *is* 1; $\theta$ *being the circular measure of the angle referred to.*

The value of $\frac{\theta}{\sin\theta}$ lies between 1 and $\frac{1}{\cos\theta}$. As $\theta$ is diminished, $\cos\theta$ approaches 1; and the smaller $\theta$ becomes, the more nearly does $\frac{1}{\cos\theta}$ approach 1. [Art. 94]

Therefore, by diminishing $\theta$ sufficiently, we can make $\frac{\theta}{\sin\theta}$ differ from 1 by less than any assignable quantity however small.

This is what is meant when it is said that '*the limit of* $\frac{\theta}{\sin\theta}$, *when* $\theta=0$, *is* 1.'

291. The student must notice carefully that $\theta$ here is the **number of radians** in the angle referred to.*

*Example. Prove that the limit of* $\frac{n}{\sin n^0}$, *when* $n=0$, *is* $\frac{180}{\pi}$.

Let $\theta$ radians $=n^0$, then

$$\frac{\theta}{\pi}=\frac{n}{180},\quad \therefore n=\frac{180\theta}{\pi},\quad \therefore \frac{n}{\sin n^0}=\frac{180}{\pi}\cdot\frac{\theta}{\sin\theta}.$$

When $n$ is diminished, $\theta$ is diminished also, and the limit of $\frac{\theta}{\sin\theta}$, when $\theta=0$, is 1. Therefore the limit of $\frac{n}{\sin n^0}$ $\left(\text{which} =\frac{180}{\pi}\cdot\frac{\theta}{\sin\theta}\right)$, when $n=0$, is $\frac{180}{\pi}$.

---

* It is on this account that a radian is used as a unit of angle (Art. 62).

292. *To prove that if $\theta$ be the circular measure of a positive angle less than a right angle,* sin $\theta$ *lies between $\theta$ and $\theta - \frac{1}{4}\theta^3$.*

It has been proved that $\sin\theta$ is less than $\theta$ ......... (i).

And that $\theta$ is less than $\tan\theta$.

Therefore $\frac{\theta}{2}$ is less than $\tan\frac{\theta}{2}$; or, $\frac{\theta}{2}\cos\frac{\theta}{2}$ is less than $\sin\frac{\theta}{2}$.

Now, $\sin\theta = 2\sin\frac{\theta}{2}\,.\,\cos\frac{\theta}{2}$; [Art. 166]

$$\therefore \sin\theta \text{ is greater than } 2\left(\frac{\theta}{2}\cos\frac{\theta}{2}\right)\cos\frac{\theta}{2},$$

i.e. greater than $\theta\cos^2\frac{\theta}{2}$, i.e. greater than $\theta - \theta\sin^2\frac{\theta}{2}$.

But $\left(\frac{\theta}{2}\right)^2$ is greater than $\sin^2\frac{\theta}{2}$.

Still greater therefore is $\sin\theta$ than $\theta - \frac{1}{4}\theta^3$.........(ii).

Hence, $\sin\theta$ is less than $\theta$, and greater than $\theta - \frac{\theta^3}{4}$.

293. *To find* sin 10″.

In the above, $\theta$ is the *circular measure* of the angle.

The circular measure of 10″, correct to three significant figures, is ·0000484.... [Examples X. (17).]

Let $\theta =$ ·0000484....

Then,

$$\theta - \tfrac{1}{4}\theta^3 = \theta - \tfrac{1}{4}(\cdot 0000484\ldots.)^3 = \theta - \cdot 000000000000028\ldots.$$

Hence, $\theta$ and $(\theta - \frac{1}{4}\theta^3)$ are decimal fractions which agree in their first twelve figures at least.

And since $\sin 10''$ lies *between* these fractions, therefore the first twelve decimal places of $\sin 10''$ are the same as those of the circular measure of $10''$.

Hence, if the value of $\pi$ be given to a sufficient number of decimal places, we can calculate the circular measure of $10''$, and therefore also $\sin 10''$, to 12 decimal places.

294. *To show how to construct a Table giving the Trigonometrical Ratios of angles which form an arithmetical progression having* $10''$ *for the common difference.*

In the identity

$$\sin(n+1)\alpha + \sin(n-1)\alpha = 2\sin n\alpha \,.\, \cos\alpha,$$

let $\alpha = 10''$, and suppose a table of the sines of all angles at intervals of $10''$ to have been calculated up to $n \,.\, 10''$.

Then, $\sin(n-1)\alpha$, $\sin n\alpha$ and $\cos\alpha\left[=\sqrt{1-\sin^2 10''}\right]$ are known.

Therefore by the above formula $\sin(n+1)\alpha$ can be found.

Hence, since we know $\sin 10''$, the sine of $20''$ can be found; and then the value of $\sin 30''$; and so on.

295. When the sines of angles up to $45^\circ$ have been calculated, the rest may be found from the formula,

$$\sin(45^\circ + A) - \sin(45^\circ - A) = \sqrt{2} \,.\, \sin A.$$

Also, when the sines of angles up to $60^\circ$ have been calculated, the remainder up to $90^\circ$ can be found still more easily from the formula, $\sin(60^\circ + A) - \sin(60^\circ - A) = \sin A$.

The other ratios may be found from the following:

$$\cos A = \sin(90^\circ - A), \quad \tan A = \frac{\sin A}{\cos A}, \quad \cot A = \tan(90^\circ - A),$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \quad \sec A = \operatorname{cosec}(90^\circ - A).$$

296. Since $3^\circ = 18^\circ - 15^\circ$, the sines of angles differing by $3^\circ$, or by any multiple of $3^\circ$, can be found independently. (See p. 120.) These values may be used to *test* the accuracy of the Tables calculated as above.

297. The following formulæ may be also used to test the accuracy of the Tables,

$$\sin(36^\circ+A)-\sin(36^\circ-A)-\sin(72^\circ+A)+\sin(72^\circ-A)=\sin A,$$

$$\cos(36^\circ+A)+\cos(36^\circ-A)-\cos(72^\circ+A)-\cos(72^\circ-A)=\cos A.$$

They are called **formulæ of verification.**

## On the Limit of the Visible Horizon.

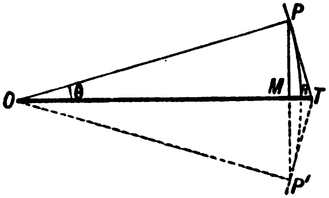
298. The surface of the sea is very nearly that of a sphere whose radius is 3957 miles.

The height of the highest mountains on the globe is less than 6 miles. Thus a point must be considered to be at a very considerable height above the surface of the sea if its height is a thousandth part of the earth's radius.

299. In the figure, let $O$ be the centre of the earth, $PRP'$ part of the surface of the sea, $T$ a point of observation, $TR$ its vertical height.

Draw $TP$, $TP'$ tangents to the earth's surface. Then $PMP'$ is parallel to the 'horizontal plane' at $R$.

The angle $TPM$ is called the **dip of the horizon at $T$.** It is the angle of depression of the most distant point on the horizon seen from $T$. It must obviously be a very small angle, since $TR$ is so small compared with $RO$.



300. If $RO$ be produced to cut the circle again in $L$, then

$$TP^2 = TR \,.\, TL \qquad \text{[Euc. III. 36]}$$

$$= TR\,(RL + TR) = 2TR \,.\, RO + TR^2.$$

But $TR$ will in general be much less than a thousandth part of $RO$, and therefore $TR^2$ will be much less than a thousandth part of $2TR \,.\, RO$.

Hence, the formula $TP^2 = 2TR \,.\, RO$, i.e. $TP^2$ = twice the earth's radius × vertical height, will give the value of $TP$ correct to at least three significant figures.

*Example. Three times the height* in feet *of the place of observation above the sea is equal to twice the square of the distance of the horizon* in miles.

Here, $TP^2 = RL \times 7914$ miles.

Let $f$ be the number of feet in $RL$, then the number of miles in $RL$ is $\frac{f}{5280}$; let $x$ be the number of miles in $TP$, then

$$x^2 = \frac{f}{5280} \times 7914 = \frac{3f}{2} \text{ nearly.} \quad \text{Q.E.D.}$$

## EXAMPLES. LXXV.

(1) Show that the limit of $\frac{n}{2} R^2 \sin \frac{2\pi}{n}$ (i.e. the area of a polygon of $n$ sides inscribed in a circle of radius $R$), when $n = \infty$ is $\pi R^2$.

(2) Prove that the limit of $nr^2 \tan \frac{\pi}{n}$, when $n = \infty$, is $\pi r^2$.

(3) Given that $\pi = 3{\cdot}141592653589793\ldots$ prove that the circular measure of $10''$ is $\cdot00004848136811\ldots$

(4) Prove that $2 \sin (72^\circ + A) - 2 \sin (72^\circ - A) = (\sqrt{5} - 1) \sin A$,
and that $2 \sin (36^\circ + A) - 2 \sin (36^\circ - A) = (\sqrt{5} + 1) \sin A$.

(5) If a mast of a ship be 150 feet high, show that the greatest distance seen from its top is 15 miles nearly.

(6) Prove that if the dip of the horizon at the top of a mountain is $1^\circ 26'$ $[= \tan^{-1} {\cdot}025]$, the mountain is about 6530 feet high.

*NOTE.* The definitions given in Arts. 75, 78 of the Trigonometrical Ratios are now used exclusively.

The NAMES *tangent*, *secant*, *sine*, were given originally to quantities defined as follows.

Let *ROP* be any angle. With centre *O* and any radius describe the arc *RP*. Draw *PM* perpendicular to *OR* and *PT* perpendicular to *OP*. (See Figure on previous page.)

Then *PR* is called an *arc*, *PT* is the *tangent of the arc PR*, *OT* is the *secant of the arc PR*, *MP* is the *sine of the arc PR*.

The name *sine* is derived from the word *sinus*. For, in the figure, *PMP′* is the string of the "bow" (*arcus*), and the string of a bow when in use is pulled to the archer's *breast*.

The *co*-tangent, *co*-secant and *co*-sine are respectively the tangent, secant and sine of the *complement* of the arc or of the angle.

The sine, tangent, etc. of the *angle* are the same as the *measures* of the sine, tangent, etc. of the *arc*, when the radius of the circle is the unit of length.

# APPENDIX.

## THE VERNIER, THE LEVEL, THE THEODOLITE, THE SEXTANT, THE MARINER'S COMPASS.

301. THE practical Surveyor[†] has to measure **distances** and **angles**, and has also to make **plans** or pictures, recording the result of his measurements.

For the measurement of distances the surveyor uses either **rods**, or **chains**, or **tapes**.

Rods used in measurement are made of *wood*, or of *metal* or sometimes (when extreme accuracy is required, as in the case of the measurement of the base line of the ordnance survey on Salisbury Plain) of *glass*.

All these instruments, when exposed to changes of temperature, are liable to change of length; hence for great accuracy, a surveyor must know the exact length of his measuring rod at all ordinary temperatures; and when making a measurement, must note the temperature of his rod at the instant of observation. The change of length caused by change of temperature is greater in a rod of *metal* than in a rod of wood. Hence *wood* is a very suitable material for measuring rods under ordinary circumstances. A *tape* made of cotton or hemp if used for measurement must be carefully protected from *moisture* by varnish or otherwise; as such tapes sensibly shrink when allowed to become damp; also, if of any considerable length, they stretch sensibly under tension.

A tape of 66 feet can be easily stretched an inch or so.

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[†] It must not be supposed that any verbal or pictorial description, such as the following, can in any way take the place of a practical explanation of the instruments themselves. A study of these figures may perhaps tell the student what to look for when he actually has the instrument in his hands.

## THE VERNIER.

302. A **vernier** is a simple instrument for increasing the accuracy of the measurement of a small distance by one significant figure.

303. **Description of a Vernier.** Suppose we have a **rule** (i.e. a measuring rod) of brass graduated† to tenths of an inch.

The vernier is a little slip of brass which slides along the rule.

This slip of brass is a little more than $1\frac{1}{10}$ inches long, and a portion of its length $1\frac{1}{10}$ inches in length is divided into *ten* equal parts, by fine scratches on the surface of the metal.

Thus the distance between each scratch and the next is $\frac{11}{100}$ of an inch or $(\frac{1}{10}+\frac{1}{100})$ of an inch; i. e. this distance exceeds the distance between two scratches on the rule, by an hundredth part of an inch.

304. **To read the Vernier.** This will be best explained by an example.

Suppose the length to be measured is ascertained to be 3 ft. 11·5 inches and a little over.

This can be ascertained by the use of the rule (or measuring rod). Now let the rule be so placed that one end exactly coincides with one extremity of the length to be measured; then the other extremity $K$ of the length to be measured will be between the scratches on the rule indicating 3 ft. 11·5 in. and 3 ft. 11·6 in. Now slide the $CD$ vernier on the rule till its extremity $D$ coincides with the extremity $K$ of the length to be measured.

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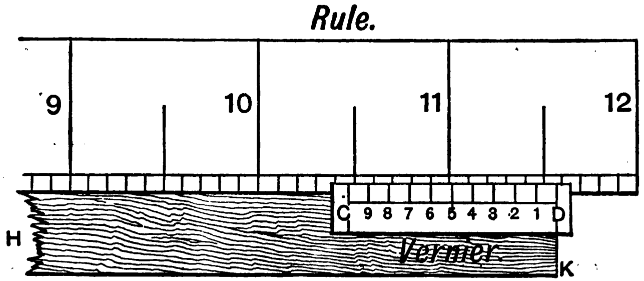
† i.e. having fine scratches upon it, each the tenth part of an inch from the next one.



It will be observed that one of the scratches on the vernier coincides with one of the scratches on the rule more nearly than any other.

Suppose this to be the scratch marked 6 on the vernier. Then the length to be measured is

3 ft. 11·56 inches nearly.

For the length exceeds 3 ft. 11·5 inches by just as much as 6 spaces on the vernier exceed 6 tenths of an inch, that is by 6 hundredths of an inch.

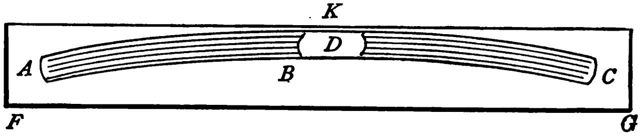
305. A vernier may be used to read the graduations of a circular arc; in which case it is made curved so as to follow the line of the arc.

306. The student should notice that the advantage gained by the use of the vernier depends on the fact that the eye is able to judge with considerable accuracy when two scratches are or are not, *coincident.*

Note. French instrument makers make their verniers $(\frac{1}{10}-\frac{1}{100})$ of an inch and divide it into ten equal parts. We leave it as an exercise to the student to discover how such a vernier would be used.

## The Level.

307. A **level** is an instrument used for ascertaining whether a given straight line is or is not horizontal.



The essential part of it consists of a glass tube $ABC$, in the form of the arc of a circle, which is closed at both ends and is nearly full of water.

The tube, being *not quite* full of water, will have a bubble $D$ in it.

This bubble, when the tube is at rest, will only rest at *the highest point* of the tube.

The tube is fixed in a case of wood or metal which is so made that when the base of it $FG$ is horizontal the highest point $K$ of the tube is visible, and the bubble can be seen at rest in it at the highest point of the arc of the tube.

This highest point is carefully marked on the tube.

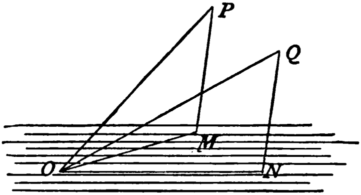
To ascertain whether any given line is horizontal it is only necessary to put the base $FG$ of the case of the 'level' in the position of that line and watch the position which the bubble takes up when the tube is kept at rest in that position.

If the bubble rests at the position marked on the tube the line is horizontal and not otherwise.

308. The student can easily purchase for himself an ordinary carpenter's level and can make experiments with it.

## THE THEODOLITE.

309. A **Theodolite** is an instrument for measuring the *horizontal* angle subtended at the position of observation by two distant visible objects.



Let $P$, $Q$ be two visible distant objects seen from a place of observation $O$.

Let $PM$, $QN$ perpendiculars be let fall from $P$ and $Q$ respectively to the horizontal plane passing through $O$.

Then the **horizontal angle** subtended at $O$ by $P$ and $Q$ is the angle $MON$ [See Examples LXXVI. (16)].

[The angle subtended at $O$ by $P$ and $Q$ is the angle $POQ$.]

310. The essential part of a Theodolite may be described as follows.

Suppose two circular brass plates to be laid one on the other so that they are concentric and both are free to turn about an axis through their centre.

Let the rim of the lower plate be graduated.

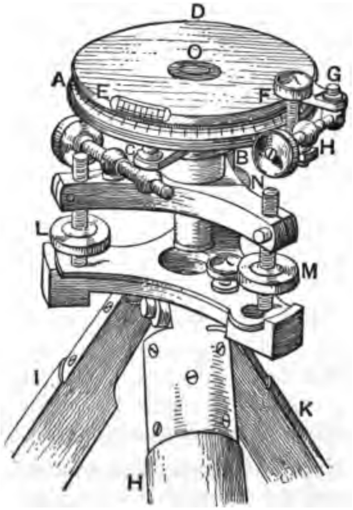
That is on the rim will be marked 360 lines at equal distances indicating degrees subtended at the centre. Each degree will be subdivided into minutes etc. according to the size of the circle and the degree of accuracy to which the instrument is to be read.

Let the rim of the upper plate have inscribed on it a vernier suitable for reading the graduations of the other rim.

Now suppose the *lower* plate can be fixed in a horizontal position and that a telescope or other means of pointing at a distant object is mounted centrally on the upper plate.

Then by first turning the upper plate till the telescope points to one distant object and reading the vernier; and again by turning the upper plate till the telescope points to another distant object and again reading the vernier, we shall obtain from the difference of these readings, the *horizontal* angle between the two distant objects.

311. The following figure is a picture of a part of a theodolite, shewing the arrangements usually made for fixing the lower plate in a horizontal position.



*EFD* is the upper plate on which is engraved a vernier at *E*.

*AB* is a portion of the rim of the lower plate.

*IHK* are the upper parts of a tripod stand whose feet are placed on the ground.

*LM* are two of the three screws which connect the head of the tripod stand with the axis round which the plates can turn.

These screws are fastened to the tripod head by a clip which is indicated near *M*.

When the tripod stand has been firmly placed on the ground so that the tripod head is fairly horizontal the plates can be made accurately horizontal by judicious turning of the three screws *L*, *M*, *N*.

The screw at *C* allows the lower plate to be slightly turned with reference to the axis and tripod head.

The arrangement *FGH* is a 'clamp' and a 'tangent screw.'

When the screw head *F* is loosened, the upper plate can be turned quite freely round its centre; so that the telescope which is mounted on it, can be turned freely through any angle.

When the screw *F* is tightened, the upper plate can only be turned with reference to the lower plate very slightly, by turning the screw head at *B* which is a tangent screw.

312. On the following page is a picture of a complete '*Transit*' **Theodolite.**

The upper plate is here shewn carrying a telescope $TT$ which is fixed to an axle of which one end $R$ is seen.

This axle is kept fixed parallel to the plate by the two pairs of upright legs which are firmly screwed to the upper plate.

The graduated circle $XYZ$ is fixed to the axis of the telescope and turns with it.

The verniers $b$ and $c$ are fixed to the upright legs.

$a$ is a small magnifying glass for reading the verniers.

$d$ is a level fixed to the telescope and parallel to its axis.

At $f$, $e$ are the clamp and tangent screw which respectively fixes the graduated circle $XYZ$ and slowly turns it.

$g$ is the end of a level fixed parallel to the plane of the upper plate.

On the top of the upper plate itself is seen a mariner's compass.

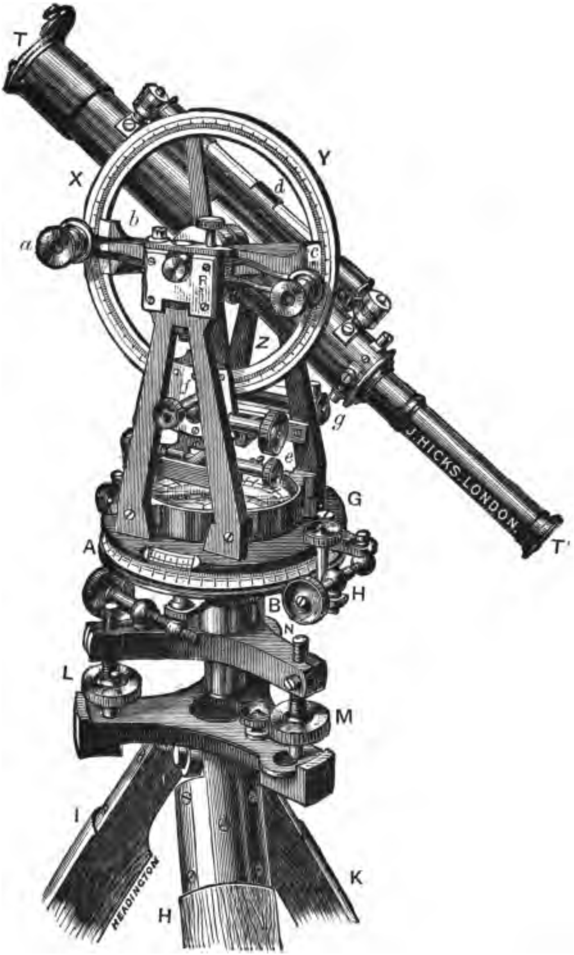
313. Suppose now that we are about to use the theodolite.

We arrange the tripod; we loosen the clamp $F$; we turn the screws. $L$, $M$, $N$ so that the plate $ABC$ is horizontal; testing this by turning the upper plate into various positions and observing the level $g$ in each position.

The instrument is now ready to make an observation.

314. **To make an observation.**

Suppose in the fig. on page v the observer is at $O$.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He turns the upper plate of the theodolite and moves the telescope until he can see $P$ through it.

He then clamps both circles $EFD$ and $XYZ$.

And by means of the tangent screws brings the point $P$ exactly into the centre of the field of view.

He then reads both verniers.

He then unclamps both circles.

And repeats the operation for the point $Q$.

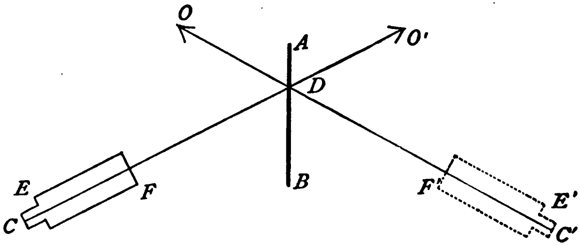
The difference between the reading of the horizontal vernier gives the *horizontal* angle $MON$.

By unclamping the vertical circle and reading the vernier when the level $d$ shews that the telescope is horizontal he can obtain each of the angles of elevation $POM$, $QON$ (from his reading of the vernier of the vertical circle when it pointed first to $P$ and then to $Q$.)

## THE SEXTANT.

315. There are three **experimental facts** connected with Optics which the student must understand who wishes to understand the principle of the sextant.

**I.** When a telescope is pointed at a plane mirror (or looking-glass) the eye sees exactly what it would if the telescope were placed on the other side of the mirror as in the following diagram.



Let the plane of the mirror be perpendicular to that of the paper. Let *ADB* be the line in which the mirror is supposed to cut the plane of the paper.

Let *EF* be a section of the telescope and *CD* the line of sight.

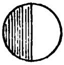
Make the angle *C′DB* equal to *CDB*.

And draw *E′F′* a dotted section of a telescope round *C′D* so that $C'D = CD$.

Then an eye placed at *C* and looking directly at the plane mirror in the direction *CO′* would see in the direction *DO*, exactly what an eye placed at *C′*, the eyepiece of a telescope whose line of sight is *C′D*, would see if the mirror were removed.

It follows that the angle *BDC* between the mirror and the axis of the telescope is **half** *ODO′*, the angle between the direction of the axis of the telescope and the direction from *D* of the object *O*.

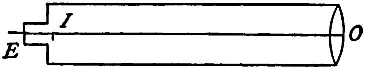
**II.** When the *half* of the object glass (or large glass) of a telescope is covered over, then an eye looking through the telescope will see exactly what it saw before except that the image will be *half* as *bright*.



The effect is very similar to the effect of looking at a picture in the first case by the light of *two* candles and in the second by the light of only *one* candle.

**III.** When looking through the eyepiece of a telescope the eye looks at an *image* (which is a small picture formed by the rays of light coming from the object looked at, inside the tube of the telescope) of the distant object at which the telescope is pointed.

The eyepiece or small glass of a telescope forms in fact a microscope with which the eye sees this *image* magnified.

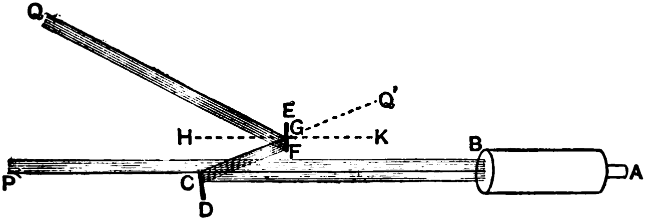


Thus in the figure, $O$ is the object-glass, $E$ is the eyepiece or microscope; $I$ is the position of the image.

$I$ is called the **focus** of the object-glass.

316. To describe the **arrangement** of a **Hadley's Sextant.**

Let the axis $AB$ of a telescope be directed to the edge $C$ of a plane mirror perpendicular to the plane of the paper whose section is $CD$.



And let $EGF$ be the section of another plane mirror arranged so that light coming from a distant object $Q$

falls on *EF* and on being reflected by the mirror falls on *CD* and on being again reflected is in the direction of the axis of the telescope *AB*.

Then an eye looking through the telescope would see the images of *two* distant objects *P* and *Q*.

These two images would be superposed.

One image that of a distant object *P* in the direction *ACD* the light from which is direct.

The other image is that of a distant object *Q* in the direction *GQ* the light from which is reflected.

The mirror *EF* is arranged so that it can turn about *G* keeping always perpendicular to the plane of the paper and the angle through which it turns can be observed.

When the mirror *EF* is turned so as to be parallel to *CD* the reflected image and the direct image would be that of the same very distant object.

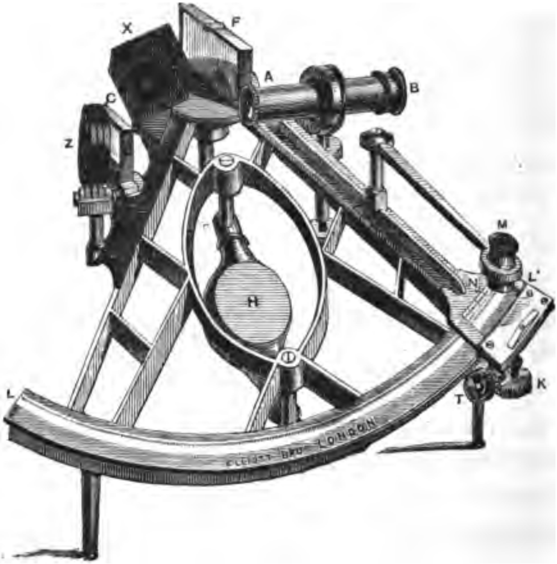
**The angle through which the mirror EF is turned, from the above position, until the image of Q is visible in the telescope is** half the angle between *CP* and *GQ*; that is, **half the angle subtended by P and Q at the observer's eye.**

[*P* and *Q* are always very distant objects such as the horizon at sea or the sun or a star.]

For, produce *GC* to *A'B'*; then the angle *PCB'* is fixed.

Draw *GK* perpendicular to *EF*; the angle turned through by *EF* in any moment, is equal to the angle turned through by *EK*; and the angle *QEC* is always double of *KEB'*; therefore the amount of turning of *EQ* relative to *EC* (which is fixed) is *double* that of *EK* relative to *EB'*; that is is *double* of the amount of turning of *EF*.

317. The following is a picture of a **Hadley's Sextant.**



*AB* is the telescope; *C* is the fixed mirror; *F* is the mirror which can be turned about an axis perpendicular to the plane of *ABCF*.

*FN* is an arm by which the mirror *F* is turned.

*LL'* is a graduated arc and *N* is a vernier for reading its graduations with the aid of the microscope *M*; *K* and *T* are the usual clamp and tangent screw; *H* is a handle for holding the instrument.

At *Z* are three blackened glasses to interpose between the light and the telescope when looking at a bright object such as the sun; each glass is on a hinge so that it can be brought into the line of sight or turned back at pleasure.

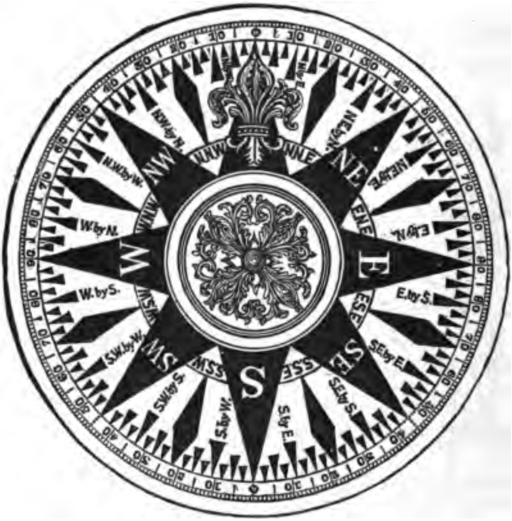
At *X* are blackened glasses which can be placed in the line of the reflected light or turned out of it at pleasure.

318. Thus, the sextant is an instrument for observing the angle subtended at the observer's eye by two distant objects. Its *peculiar advantage* is that it does not require a *steady* platform as the Theodolite does. The observer holds it in his hands, and the observation consists in his noting when he has so moved the mirror [*EF*] that the *images* of the two distant objects, whose angle he wishes to take, are superposed in the field of the telescope.

This he can do even if he cannot get the images to *rest* for any length of time in the field of view.

Accordingly the angle subtended by the edge of the sun and the horizon can with a sextant be observed with considerable accuracy by an observer standing on the deck of a ship in motion. Also, at night, the angular distance between the moon and a star can be observed under like circumstances.

319. Below we give a figure of a **Mariner's Compass.**



It consists of an ordinary magnetic compass with a card attached to the needle, this card is so arranged that when the needle is pointing along the magnetic meridian the pointer on the card is pointing due North.

The **Points** of the Compass are figured on the card.

They are **N.** = North.
**N.** by **E.** = North by East.
**N.N.E.** = North North East.
**N.E.** by **N.** = North East by North.
**N.E.** = North East.
**N.E.** by **E.** = North East by East.
**E.** by **N.** = East by North.
**E.** = East. And so on.

The angle subtended at the centre of the card by two consecutive points is $= \frac{1}{8}$ of $90^\circ = 12\frac{1}{2}^\circ$.

## EXAMPLES FOR EXERCISE. LXXVI a.

1. Define the terms sine, cotangent; and prove that if $A$ be *any* angle,
$$\sin^2 A + \cos^2 A = 1.$$
If $\tan A = \frac{3}{4}$, find $\sin A$ and $\cos A$.

2. Trace the changes in the sign and magnitude of $\cos\theta - \sec\theta$ for values of $\theta$ between 0 and $\pi$.

3. Prove geometrically that $\cos(180^\circ - A) = -\cos A$.
Find $A$ if $2\sin A = \tan A$.

4. Prove

(1) $\sin(A+B) \,.\, \sin(A-B) = \sin^2 A - \sin^2 B$;

(2) $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \dfrac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$.

5. Prove that
$$\cos^2 A - \cos A \cos(60^\circ + A) + \sin^2(30^\circ - A) = \tfrac{3}{4}.$$

6. Find the greatest side of the triangle of which one side is 2183 feet and the adjacent angles are $78^\circ\,14'$ and $71^\circ\,24'$.

$\log 2183 = 3{\cdot}3390537$, $\log 42274 = 4{\cdot}6260733$,
$L\sin 78^\circ\,14' = 9{\cdot}9907766$, $\log 42275 = 4{\cdot}6260836$.
$L\sin 30^\circ\,22' = 9{\cdot}7037486$,

---

7. Express the other trigonometrical ratios in terms of the cosine.

8. Prove
$$\sin(180 + A) = -\sin A;$$
$$\tan(90 + A) = -\cot A.$$

9. Write down the sines of all the angles which are multiples of $30^\circ$ and less than $360^\circ$.

10. Prove
$$\tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}.$$

11. If $\tan A + \sec A = 2$, prove that $\sin A = \frac{3}{5}$, when $A$ is less than $90^\circ$.

If $\sin A = \frac{4}{5}$, prove that $\tan A + \sec A = 3$, when $A$ is less than $90^\circ$.

12. The length of the greatest side of a triangle is 1035·43 feet, and the three angles are $44^\circ$, $66^\circ$, and $70^\circ$. Solve the triangle, having given

$$L\sin 44^\circ = 9{\cdot}8417713, \qquad L\sin 66^\circ = 9{\cdot}9607302,$$
$$L\sin 70^\circ = 9{\cdot}9729858, \qquad \log 1035{\cdot}43 = 3{\cdot}0151212,$$
$$\log 765432 = 5{\cdot}8839067, \qquad \log 10066 = 4{\cdot}0028656.$$

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13. Express the other trigonometrical ratios in terms of the cotangent.

14. Prove that
$$\cos(180^\circ - A) = -\cos A;$$
$$\operatorname{cosec}(180^\circ + A) = -\operatorname{cosec} A.$$

15. Write down the tangents of all the angles which are multiples of $30^\circ$ and less than $360^\circ$.

16. If $\tan A + \sec A = 3$, prove that $\sin A = \frac{4}{5}$ when $A$ is less than $90^\circ$.

If $\sin A = \frac{3}{5}$, prove that $\tan A + \sec A = 2$, when $A$ is less than $90^\circ$.

17. Find the sines of the three angles of the triangle whose sides are 193, 194, and 195 feet.

18. Investigate the following formulæ:

(1) $$\cos\frac{3A}{2} = (2\cos A - 1)\cos\tfrac{1}{2}A;$$

(2) $$\cos\theta - \cos(\theta + \delta) = \sin\theta\sin\delta\,(1 + \cot\theta\tan\tfrac{1}{2}\delta).$$

---

19. Define the secant of an angle.

Prove the formula $\dfrac{1}{\sec^2 A} + \dfrac{1}{\operatorname{cosec}^2 A} = 1.$

If $\sin A = \frac{1}{3}$, find $\sec A$.

20. Trace the changes in sign and magnitude of
$$\frac{\sin\theta + \cos\theta}{\sin\theta - \cos\theta}, \text{ as } \theta \text{ changes from } \pi \text{ to } 2\pi.$$

21. Find a formula to include all angles that have the same cotangent as the angle $\alpha$.
Solve the equation $\tan\theta = \cot\theta$.

22. Prove the formula to express the cosine of the sum of two angles in terms of the sines and cosines of those angles.
Express $\cos 5\alpha$ in terms of $\cos\alpha$.

**23.** Prove the formula

$$2\sin\tfrac{1}{2}A = \pm\sqrt{(1+\sin A)} \pm \sqrt{(1-\sin A)}.$$

Account for the double signs in this formula, and examine which signs must be taken if $A$ be an angle between $540^\circ$ and $630^\circ$.

**24.** A ring 10 inches in diameter is suspended from a point 1 foot above its centre by six equal strings attached to its circumference at equal intervals; find the cosine of the angle between two consecutive strings.

---

**25.** Define $1^\circ$. Assuming that $\frac{22}{7}$ is the circular measure of two right angles, express the angle $A^\circ$ in circular measure.

Find the number of degrees in the angle whose circular measure is ·1.

**26.** Find the trigonometrical ratios of the angle whose cosine is $\frac{3}{5}$.

**27.** Prove that

$$(1)\quad \cos(180^\circ + A) = \cos(180^\circ - A);$$
$$(2)\quad \tan(90^\circ + A) = \cot(180^\circ - A).$$

**28.** Prove $\sin x(2\cos x - 1) = 2\sin\frac{x}{2}\cos\frac{3x}{2}$.

**29.** Trace the changes in sign and magnitude of

$$\frac{2\sin\theta - \sin 2\theta}{2\sin\theta + \sin 2\theta},$$

as $\theta$ changes from 0 to $2\pi$.

**30.** If the angle opposite the side $a$ be $60^\circ$, and if $b$, $c$ be the remaining sides of the triangle, prove that

$$(a+b+c)(b+c-a) = 3bc.$$

---

**31.** Assuming $\frac{22}{7}$ to be the circular measure of two right angles, express in degrees the angle whose circular measure is $\theta$. Find the number of degrees in an angle whose circular measure is $\frac{1}{3}$.

**32.** Prove that $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$.

**33.** Prove $\sin x(2\cos x + 1) = 2\cos\frac{x}{2}\sin\frac{3x}{2}$.

**34.** Trace the changes in sign and magnitude of

$$\frac{\sin\theta + 2\sin\frac{1}{2}\theta}{\sin\theta - 2\sin\frac{1}{2}\theta},$$

as $\theta$ changes from 0 to $2\pi$.

35. If $(\sin A+\sin B+\sin C)(\sin A+\sin B-\sin C)=3\sin A\sin B$, and $A+B+C=180^\circ$, prove that $C=60^\circ$.

36. Given $A=18^\circ$, $B=144^\circ$, and $b=1$, solve the triangle.

---

37. Give the trigonometrical definition of an angle.

What angle does the minute-hand of a clock describe between twelve o'clock and 20 minutes to four?

38. Express the cosine and the tangent of an angle in terms of the sine.

The angle $A$ is greater than $90^\circ$ but less than $180^\circ$, and $\sin A=\frac{1}{3}$. Find $\cos A$.

39. Find an expression for all the values of $\theta$ for which

$$\cos\theta+\cos 2\theta=0.$$

40. If in a triangle $a\cos A=b\cos B$, the triangle will be either isosceles or right-angled.

41. The sides are 1 foot and $\sqrt{3}$ feet respectively, and the angle opposite to the shorter side is $30^\circ$; solve the triangle.

42. The sides of a triangle are 2, 3, 4. Find the greatest angle, having given

$$\begin{aligned}\log 2&=\cdot 3010300,\\ \log 3&=\cdot 4771213,\\ L\tan 52^\circ.15'&=10\cdot 1111004,\\ L\tan 52^\circ.14'&=10\cdot 1108395.\end{aligned}$$

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43. Distinguish between Euclid's definition of an angle and the trigonometrical definition.

What angle does the minute-hand of a clock describe between half-past four and a quarter-past six?

44. Express the sine and the cosine of an angle in terms of the tangent.

The angle $A$ is greater than $180^\circ$ but less than $270^\circ$, and $\tan A=\frac{1}{3}$. Find $\sin A$.

45. Prove (i) $\sin 2A=\dfrac{2\cot A}{1+\cot^2 A}$.

(ii) Show that if $A+B+C=90^\circ$,

$$\sin 2A+\sin 2B+\sin 2C=4\cos A\cos B\cos C.$$

46. Find an expression for all the values of $\theta$ for which

$$\sin\theta+\sin 2\theta=0.$$

47. If in a triangle $b\cos A=a\cos B$, show that the triangle is isosceles.

48. The sides are 1 foot and $\sqrt{2}$ feet respectively, and the angle opposite to the shorter side is $30^\circ$; solve the triangle.

---

49. Express in degrees, minutes, etc. (1) the angle whose circular measure is $\frac{1}{30}\pi$; (2) the angle whose circular measure is 5.

If the angle subtended at the centre of a circle by the side of a regular heptagon be the unit of angular measurement, by what number is an angle of $45^\circ$ represented?

50. Prove that

$$(\sin 30^\circ + \cos 30^\circ)(\sin 120^\circ + \cos 120^\circ) = \sin 30^\circ.$$

51. Prove the formulæ:

(1) $\cos^2(\alpha+\beta) - \sin^2\alpha = \cos\beta\cos(2\alpha+\beta)$;

(2) $1 + \cot\alpha\cot\frac{1}{2}\alpha = \operatorname{cosec}\alpha\cot\frac{1}{2}\alpha$.

52. Solve the equations:

(1) $5\tan^2 x - \sec^2 x = 11$; (2) $\sin 5\theta - \sin 3\theta = \sqrt{2}\,.\cos 4\theta$.

53. Two sides of a triangle are 10 feet and 15 feet in length, and the angle between them is $30^\circ$. What is its area?

54. Given that

$$\sin 40^\circ\, 29' = 0{\cdot}6492268, \quad \sin 40^\circ\, 30' = 0{\cdot}6494480,$$

find $\sin^{-1}(0{\cdot}6493000)$.

---

55. Express in circular measure (1) $10'$, (2) $\frac{1}{5}$ of a right angle.

If the angle subtended at the centre of a circle by the side of a regular pentagon be the unit of angular measurement, by what number is a right angle represented?

56. If $\sec\alpha = 7$, find $\tan\alpha$ and $\operatorname{cosec}\alpha$.

57. Prove the formulæ:

(1) $\cos^2(\alpha-\beta) - \sin^2(\alpha+\beta) = \cos 2\alpha\cos 2\beta$;

(2) $1 + \tan\alpha\tan\frac{1}{2}\alpha = \sec\alpha$.

58. Solve the equations:

(1) $5\tan^2 x + \sec^2 x = 7$; (2) $\cos 5\theta + \cos 3\theta = \sqrt{2}\,.\cos 4\theta$.

59. The lengths of the sides of a triangle are 3 feet, 5 feet, and 6 feet. What is its area?

60. Given that

$$\sin 38^\circ\, 25' = 0{\cdot}6213757, \quad \sin 38^\circ\, 26' = 0{\cdot}6216036,$$

find $\sin^{-1}(0{\cdot}6215000)$.

---

61. Which is greater, $76^g$ or the angle whose circular measure is $1{\cdot}2$?

62. Determine geometrically $\cos 30^0$ and $\cos 45^0$.

If $\sin A$ be the arithmetic mean between $\sin B$ and $\cos B$, then

$$\cos 2A = \cos^2 (B + 45^0).$$

63. Establish the following relations:

(1) $\tan^2 A - \sin^2 A = \tan^2 A \sin^2 A$;

(2) $\cot A - \cot 2A = \operatorname{cosec} 2A$;

(3) $\dfrac{\sin (x + 3y) + \sin (3x + y)}{\sin 2x + \sin 2y} = 2 \cos (x + y)$.

64. Show that for certain values of the angles

$$2 \sin \tfrac{1}{2}A = \sqrt{1 + \sin A} - \sqrt{1 - \sin A}.$$

Is the formula true when $A = 240^0$? If not, how must it be modified?

65. Prove that $\sin (A + B) = \sin A \cos B + \cos A \sin B$, and deduce the expression for $\cos (A + B)$.

Show that

$$\sin A \cos (B + C) - \sin B \cos (A + C) = \sin (A - B) \cos C.$$

66. One side of a triangular lawn is 102 feet long, its inclinations to the other sides being $70^0\,30'$, $78^0\,10'$ respectively. Determine the other sides and the area. $L \sin 70^0\,30' = 9{\cdot}974$, $\log 102 = 2{\cdot}009$, $L \sin 78^0\,10' = 9{\cdot}990$, $\log 185 = 2{\cdot}267$, $L \sin 31^0\,20' = 9{\cdot}716$, $\log 192 = 2{\cdot}283$, $\log 2 = {\cdot}301$, $\log 9234 = 3{\cdot}965$.

67. Which is greater, $126^0$ or the angle whose circular measure is $2{\cdot}3$?

68. Establish the following relations:

(1) $\cot^2 A - \cos^2 A = \cot^2 A \cos^2 A$;

(2) $\tan A + \cot 2A = \operatorname{cosec} 2A$;

(3) $\dfrac{\cos (x - 3y) - \cos (3x - y)}{\sin 2x + \sin 2y} = 2 \sin (x - y)$.

69. Show that for certain values of the angles

$$2 \cos \tfrac{1}{2}A = \sqrt{1 + \sin A} + \sqrt{1 - \sin A}.$$

Is the formula true when $A = 300^0$? If not, how must it be modified?

70. Prove that $\sin 30^0 + \sin 120^0 = \sqrt{2} \cos 15^0$.

71. Establish the identities:

(1) $1 + \cos A + \sin A = \sqrt{2 (1 + \cos A)(1 + \sin A)}$;

(2) $\operatorname{cosec} 2A = \dfrac{\operatorname{cosec}^2 A}{2 \sqrt{\operatorname{cosec}^2 A - 1}}$;

(8) $\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}-\sin\frac{6\pi}{7}=4\sin\frac{\pi}{7}\sin\frac{3\pi}{7}\sin\frac{5\pi}{7}$.

72. The sides of a triangular lawn are 102, 185, and 192 feet in length, the smallest angle being approximately $31^\circ 20'$. Find its other angles and its area.

$\log 102=2\cdot009$, $L\sin 31^\circ 20'=9\cdot716$,
$\log 185=2\cdot267$, $L\sin 70^\circ 30'=9\cdot974$,
$\log 192=2\cdot283$, $L\sin 78^\circ 10'=9\cdot990$,
$\log 2=\cdot301$, $\log 9234=3\cdot965$.

---

73. If the circumference of a circle be divided into five parts in arithmetical progression, the greatest part being six times the least, express in radians the angle each subtends at the centre.

74. Define the sine of an angle, wording your definition so as to include angles of any magnitude.

Prove that $\sin(90^\circ+A)=\cos A$,
and $\cos(90^\circ+A)=-\sin A$,
and by means of these deduce the formulæ

$$\sin(180^\circ+A)=-\sin A,\quad \cos(180^\circ+A)=-\cos A.$$

75. Prove the formulæ:

(1) $\cot^2 A=\operatorname{cosec}^2 A-1$;
(2) $\cot^4 A+\cot^2 A=\operatorname{cosec}^4 A-\operatorname{cosec}^2 A$.

Verify (2) when $A=30^\circ$.

76. Show that all angles satisfying the equation

$$\cos\theta=\cos\alpha$$

are included in the formula $\theta=2n\pi\pm\alpha$.

Solve completely the equation $2\cos^3\theta+\sin^2\theta-1=0$.

77. If $\sin B$ be the geometric mean between $\sin A$ and $\cos A$, then

$$\cos 2B=2\cos^2(A+45^\circ).$$

78. The lengths of two of the sides of a triangle are 1 foot and $\sqrt{2}$ feet respectively, the angle opposite the shorter side is $30^\circ$. Prove that there are two triangles which satisfy these conditions; find their angles, and show that their areas are in the ratio $\sqrt{3}+1:\sqrt{3}-1$.

---

79. If the circumference of a circle be divided into six parts in arithmetical progression, the greatest being six times the least, express in radians the angle each subtends at the centre.

80. Define the tangent of an angle, wording your definition so as to include angles of any magnitude.

Prove that $\tan(90^\circ+A)=-\cot A$, and by means of this formula deduce the formula $\tan(180^\circ+A)=\tan A$.

**81.** Show that all angles satisfying the equation

$$\tan\theta = \tan\alpha$$

are included in the formula $\theta = n\pi + \alpha$.

Solve completely the equation $\sec^3\theta - 2\tan^2\theta = 2$.

**82.** Prove that $\cos(A+B) = \cos A\cos B - \sin A\sin B$, and deduce the expression for $\sin(A+B)$.

Show that

$$\cos A\cos(B+C) - \cos B\cos(A+C) = \sin(A-B)\sin C.$$

**83.** Establish the identities:

(1) $1 + \cos A - \sin A = \sqrt{2(1+\cos A)(1-\sin A)}$;

(2) $\sec 2A = \dfrac{\sec^2 A}{2 - \sec^2 A}$;

(3) $\cos\dfrac{2\pi}{7} + \cos\dfrac{4\pi}{7} + \cos\dfrac{6\pi}{7} + 4\cos\dfrac{\pi}{7}\cos\dfrac{3\pi}{7}\cos\dfrac{5\pi}{7} + 1 = 0.$

**84.** Two adjacent sides of a parallelogram 5 in. and 3 in. long respectively, include an angle of 60°. Find the lengths of the two diagonals and the area of the figure.

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**85.** Investigate the following formulæ:

(1) $\sin\dfrac{3A}{2} = (1 + 2\cos A)\sin\tfrac{1}{2}A$;

(2) $\sin(\theta+\delta) - \sin\theta = \cos\theta\sin\delta(1 - \tan\theta\tan\tfrac{1}{2}\delta)$.

**86.** Prove that

(1) $\sin 10^\circ + \sin 50^\circ = \sin 70^\circ$;

(2) $\sqrt{3} + \tan 40^\circ + \tan 80^\circ = \sqrt{3}\tan 40^\circ\tan 80^\circ$;

(3) if $A + B + C = 180^\circ$,

$$\frac{\sin A - \sin B\cos C}{\cos B} = \frac{\sin B - \sin A\cos C}{\cos A}.$$

**87.** Find the value of $\sin 18^\circ$, and deduce that

$$4\sin 18^\circ\cos 36^\circ = 1.$$

**88.** The length of one side of a triangle is 1006·62 feet and the adjacent angles are 44° and 70°. Solve the triangle, having given

$L\sin 44^\circ = 9{\cdot}8417713$, $L\sin 70^\circ = 9{\cdot}9729858$,
$L\sin 66^\circ = 9{\cdot}9607302$, $\log 1006{\cdot}62 = 3{\cdot}0028656$,
$\log 7654321 = 6{\cdot}8839067$, $\log 103543 = 5{\cdot}0151212$.

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89. Find the length of the arc of a circle whose radius is 8 feet which subtends at the centre an angle of $50^0$, having given

$$\pi = 3\cdot1416.$$

90. Prove that $\sin A = -\sin(A - 180^0)$.
Find the sines of $30^0$ and $2010^0$.

91. Investigate an expression for all angles which have a given cosecant.

Write down the general value of $\operatorname{cosec}^{-1}(-\sqrt{2})$.

92. Prove that

(1) $\cos^2 A + \cos^2 B - 2\cos A \cos B \cos(A+B) = \sin^2(A+B)$;
(2) $\cos^2 A + \sin^2 A \cos 2B = \cos^2 B + \sin^2 B \cos 2A$.

93. Prove that in any triangle

$$a^2 \cos 2B + b^2 \cos 2A = a^2 + b^2 - 4ab \sin A \sin B.$$

94. If $a = 123$, $B = 29^0 17'$, $C = 135^0$, find $c$, having given

$\log 123 = 2\cdot0899051$, $\log 2 = \cdot3010300$,
$\log 3211 = 3\cdot5066403$, diff. for $1 = 1352$.
$L \sin 15^0 43' = 9\cdot4327777$.

---

95. Define the unit of circular measure, and prove that it is an invariable angle.

If an arc of 12 feet subtend at the centre of a circle an angle of $50^0$, what is the radius of the circle, $\pi$ being equal to $3\cdot1416$?

96. Express the cosine and cotangent in terms of the cosecant.

If $\cot A + \operatorname{cosec} A = 5$, find $\cos A$.

97. Investigate an expression for all angles which have a given secant.

Write down the general value of $\sec^{-1}(-2)$.

98. Prove that

(1) $\sin^2 A + \sin^2 B + 2\sin A \sin B \cos(A+B) = \sin^2(A+B)$;
(2) $\sin^2 A - \cos^2 A \cos 2B = \sin^2 B - \cos^2 B \cos 2A$.

99. Prove that $2\cos\frac{1}{2}A = \pm\sqrt{1+\sin A} \pm \sqrt{1-\sin A}$, and determine the proper signs of the roots when $A = 2400^0$.

100. Prove that in any triangle

$$\frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2} = \frac{1}{a^2} - \frac{1}{b^2}.$$

---

101. Prove (i) $2\cot 2A = \cot A - \tan A$,
(ii) $\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{5}{13} = \sin^{-1}\frac{16}{65}$,
(iii) $\cot(A + 15^0) - \tan(A - 15^0) = \dfrac{4\cos 2A}{2\sin 2A + 1}$

102. Solve the equations

$$\cos(2x+3y)=\tfrac{1}{2},\ \cos(3x+2y)=\tfrac{1}{2}\sqrt{3}.$$

103. Trace the changes in the sign and magnitude of $\sin(\pi\sin x)$ as $x$ increases from 0 to $2\pi$.

104. Solve the equation

$$\tan^{-1}x+\tan^{-1}(1-x)=2\tan^{-1}\sqrt{x-x^2}.$$

105. A ship sailing due north observes two lighthouses bearing respectively N.E. and N.N.E.; after sailing 20 miles the lighthouses are seen to be in a line due east; find the distance between the lighthouses in miles accurately to four places of decimals; having given,

$\log 2=\cdot3010300$, $L\tan 22^\circ30'=9\cdot6172243$, $\log 11\cdot715=1\cdot0687423$, and $\log 11\cdot716=1\cdot0687794$.

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106. Prove (i) $\tan\frac{\theta}{2}=\sqrt{\left(\frac{1-\cos\theta}{1+\cos\theta}\right)}$,

(ii) $\frac{\tan 5\theta+\tan 3\theta}{\tan 5\theta-\tan 3\theta}=4\cos 2\theta\cos 4\theta$,

(iii) $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}=\sin^{-1}\frac{77}{85}$.

107. Solve the equation $2\sin^2\theta-(1+\sqrt{3})\sin 2\theta+2\sqrt{3}\cos^2\theta=0$.

108. Show that in any triangle

(i) $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0$,

(ii) $c(\cos A+\cos B)=2(a+b)\sin^2\frac{1}{2}C$.

109. Prove that if $D$, $E$, $F$ are the feet of the perpendiculars from $A$, $B$, $C$ upon the opposite sides of the triangle $ABC$ the diameters of the circumscribing circles of the triangles $AEF$, $BDF$, $CDE$ are $a\cot A$, $b\cot B$, $c\cot C$ respectively.

110. A man who is walking on a horizontal plane towards a tower observes that at a certain point the elevation of the top of the tower is 10° and after going 50 yds. nearer to the tower the elevation is 15°; find the height of the tower in yards to 4 places of decimals, having given $L\sin 15^\circ=9\cdot4129962$, $L\cos 5^\circ=9\cdot9983442$,

$\log 25\cdot783=1\cdot4113334$, $\log 25\cdot784=1\cdot4113503$.

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111. Prove that when $\sin A$ is a geometric mean between $\sin B$ and $\cos B$, then $\cos 2A=2\sin(45^\circ-B)\cos(45^\circ+B)$.

112. Prove (i) $\sin A(\cos 2A+\cos 4A+\cos 6A)=\sin 3A\cos 4A$,

(ii) $2\sin^{-1}\frac{1}{2}=\cos^{-1}\frac{1}{2}$.

113. In any quadrilateral $ABCD$ prove that

$$AB\cos A-BC\cos(A+B)+CD\cos D=AD.$$

114. Prove that in any triangle

$$\tan 2A + \tan 2B + \tan 2C = \tan 2A \tan 2B \tan 2C,$$

hence show that if $xyz$ are numbers such that

$$x + y + z = xyz,$$

then $$x(1-y^2)(1-z^2) + y(1-z^2)(1-x^2) + z(1-x^2)(1-y^2) = 4xyz.$$

115. From each of two ships a mile apart the angle is observed which is subtended by the other ship and a beacon on shore, these angles are $52^\circ\, 25'\, 15''$ and $75^\circ\, 9'\, 30''$; find the distances of the beacon from each of the ships, having given $L \sin 75^\circ\, 9'\, 30'' = 9{\cdot}9852635$, $L \sin 52^\circ\, 25'\, 15'' = 9{\cdot}8990055$, $\log 1{\cdot}2197 = {\cdot}0862530$,

$$\log 1{\cdot}2198 = {\cdot}0862886.$$

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116. The cosines of two angles of a triangle are $\frac{3}{5}$ and $\frac{12}{13}$ respectively; find all the trigonometrical ratios of the third angle.

117. Prove (i) $$\frac{\sin A + 2\sin 3A + \sin 5A}{\cos A - 2\cos 3A + \cos 5A} = \frac{4\sin A - 3\operatorname{cosec} A}{4\cos A - 3\sec A},$$

(ii) $$\cot^{-1}\tfrac{1}{3} = \cot^{-1} 3 + \cot^{-1}\tfrac{3}{4}.$$

118. The lengths of the side of a triangle are 242, 1212 and 1450 yards; show that the area is 6 acres.

119. Prove that $ad_1^2 + bd_2^2 + cd_3^2 = abc$, where $d_1$, $d_2$, $d_3$ are the distances of the centre of the inscribed circle from $A$, $B$, $C$.

120. A man observes that when he has walked $c$ feet up an inclined plane, the angular depression of an object in the horizontal plane through the foot of the slope is $\alpha$; and that when he has walked a further distance $c$ feet the angular depression of the same object is $\beta$; show that the inclination of the slope to the horizon is

$$\cot^{-1}(2\cot\beta - \cot\alpha).$$

---

121. Prove $\tan^{-1}(\tfrac{1}{2}\tan 2A) + \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A) = 0$.

122. The general solution of $\tan(45 - \tfrac{1}{2}x) + \cot(45^\circ - \tfrac{1}{2}x) = 4$ is $(6n \pm 1)\, 60^\circ$, $n$ being any integer.

123. Prove that $\sqrt{(1 + \sin A)} = 1 + 2\sin\tfrac{1}{4}A\,\sqrt{(1 - \sin\tfrac{1}{2}A)}$, when $A$ lies between $-90^\circ$ and $180^\circ$.

124. If $$a\sin A + b\sin B + c\sin C = 0,$$

and $$a\cos A + b\cos B + c\cos C = 0,$$

then $$a : b : c = \sin(B - C) : \sin(C - A) : \sin(A - B).$$

125. A man ascends a mountain by a direct course, the inclination of his path to the horizon being at first $\alpha$ and afterwards changing suddenly to $\beta$, which continues to the summit; the mountain being $a$ feet high and the angle of elevation of the summit from the starting point being $\gamma$, show that the length of the ascent is

$$a\frac{\cos\{\frac{1}{2}(\alpha+\beta)-\gamma\}}{\sin\gamma\cos\frac{1}{2}(\beta-\alpha)}.$$

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126. $\operatorname{cosec} A \operatorname{cosec} 2A + \operatorname{cosec} 2A \operatorname{cosec} 3A = \operatorname{cosec} A (\cot A - \cot 3A)$.

127. Solve the equation

$$\cos 3\theta + \cos 5\theta + \sqrt{2}(\cos\theta + \sin\theta)\cos\theta = 0.$$

128. In any triangle

$$4\sin A\sin B\sin^2\tfrac{1}{2}C = (\sin B + \sin C - \sin A)(\sin C + \sin A - \sin B).$$

129. Find (without tables) to 3 places of decimals the numbers of which $1{\cdot}5$, $\cdot\dot{3}$ and $1{\cdot}\dot{3}$ are the common logarithms.

130. Two chords diverging from the same point on the circumference of a circle are to each other as the sines of the angles they respectively make with the tangent at that point.

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131. Prove that there are eleven, and only eleven, pairs of regular polygons which are such that the number of degrees in an angle of one of them = the number of grades in an angle of the other; and that there are only four pairs when the number is an integer.

132. $$\cos 11A + 3\cos 9A + 3\cos 7A + \cos 5A \\ = 16\cos^3 A\cos(4A + \tfrac{1}{4}\pi)\cos(4A - \tfrac{1}{4}\pi).$$

133. If $\sin^{-1}x + \sin^{-1}\frac{1}{2}x = \frac{1}{4}\pi$, then $x = \sqrt{\{\frac{2}{17}(5 - 2\sqrt{2})\}}$.

134. Transform $\sqrt{(2\sec A)} = \frac{1}{5}\sqrt[3]{(\cos^2 B\operatorname{cosec} C)}$ into an equation between tabular logarithms.

135. The distances of $A$ one of the angular points of a regular octagon $ABCDEFGH$ from the sides $BC$, $CD$, $DE$ respectively are as

$$1 : 1 + \sqrt{2} : 2 + \sqrt{2}.$$

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136. $$\sin(2\alpha+\beta+\gamma)\sin(\beta-\gamma) + \sin(\alpha+2\beta+\gamma)\sin(\gamma-\alpha) \\ + \sin(\alpha+\beta+2\gamma)\sin(\alpha-\beta) = 0.$$

137. If $\log\frac{1025}{1024} = \alpha$ and $\log 2 = \beta$, then $\log 4100 = \alpha + 12\beta$.

138. $\sin 81^\circ - \sin 39^\circ - \sin 21^\circ + \sin 99^\circ = \sin 90^g$.

139. Solve $\sin(n+1)\theta + \sin(n-1)\theta = \sin 2\theta$.

140. $ABCD$ is a quadrilateral and $AB$, $DC$ produced meet in $E$; the angle $DAC = DBC = \alpha$; the angle $CAB = \beta$; the angle $CBE = \gamma$; prove that
$$BE = AB\,\frac{\sin\beta\sin(\alpha+\beta)}{\sin(\gamma-\beta)\sin(\alpha+\beta+\gamma)}.$$

---

141. $$\cos 2A + \sin 2B = 2\sin\{\tfrac{1}{4}\pi - (A-B)\}\cos\{\tfrac{1}{4}\pi - (A+B)\}$$
$$\cos 2A - \sin 2B = 2\sin\{\tfrac{1}{4}\pi - (A+B)\}\cos\{\tfrac{1}{4}\pi - (A-B)\}.$$

142. (i) $\cos 55^\circ + \cos 65^\circ + \cos 175^\circ = 0$,
(ii) $\sin^2 24^\circ - \sin^2 6^\circ = \frac{1}{8}(\sqrt{5}-1)$.

143. If $A+B+C+D = 180^\circ$, then
$$\cos 2A - \cos 2B + \cos 2C - \cos 2D = 4\sin(B+C)\cos(C+A)\sin(A+B).$$

144. Given $\log 35 = a$, $\log 325 = b$, $\log 245 = c$,
find $$\log 5,\ \log 7 \text{ and } \log 13.$$

145. A train is going due East at the rate of 24 miles an hour; when will it be 18 miles distant from a town which is on the N.E. of the train at a distance of 24 miles?

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146. If $4\sin A\tan(A-B) + 3\sec A = 4\sin^2 A\sec A$,
then $$\tan A\tan B = 3.$$

147. If $$\tan(\pi\cot\theta) = \cot(\pi\tan\theta),$$
then $$4\tan\theta = 2n+1 \pm \sqrt{(4n^2+4n-15)}.$$

148. If $$x = \cos 2\alpha + \cos\alpha,\quad y = \sin 2\alpha + \sin\alpha,$$
then $$2x = (x^2+y^2)^2 - 3(x^2+y^2).$$

149. In any triangle $2R\sin C = b\cos A + \sqrt{(a^2 - b^2\sin^2 A)}$.

150. In a triangle right-angled at $C$
$$\tan^{-1}\frac{a}{b+c} + \tan^{-1}\frac{b}{a+c} = \frac{\pi}{4}.$$

---

151. One angle of a quadrilateral is $60^\circ$, another $50^g$, another $\frac{3}{4}\pi$; express all four angles in degrees.

152. If $\sin^{-1} m + \sin^{-1} n = \frac{1}{2}\pi$, prove that $\sin^{-1} m = \cos^{-1} n$.

153. If $\cos\alpha = \dfrac{\cos\beta - e}{1 - e\cos\beta}$, then $\tan\frac{1}{2}\alpha = \sqrt{\left(\dfrac{1+e}{1-e}\right)}\tan\frac{1}{2}\beta$.

154. $(\cos\theta + \sin\theta)(\cos 2\theta + \sin 2\theta) = \cos\theta + \cos(3\theta - \frac{1}{2}\pi)$.

155. In any triangle $a^2b^2c^2 - 4b^2S^2$; $a^2b^2c^2\cos A - 4bcS^2$ and $a^2b^2c^2 - 4c^2S^2$ are in G. P.

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156. Simplify $\cos^2(A+B)+\cos^2(A-B)-\cos 2A\cos 2B$.

157. With two units differing by $10^0$ the measure of an angle is as 3 is to 4; find the units.

158. $$\cos 7^0\, 30' = \tfrac{1}{4}(-1+\sqrt{2}+\sqrt{3})\sqrt{(2+\sqrt{2})}.$$

159. In any triangle
$$\frac{\sin\frac{1}{2}A+\cos\frac{1}{2}B-\sin\frac{1}{2}C}{\sin\frac{1}{2}A+\cos\frac{1}{2}C-\sin\frac{1}{2}B}=\frac{1+\tan\frac{1}{4}B}{1+\tan\frac{1}{4}C}.$$

160. In any triangle $a^2$, $b^2-c^2$, $b^2+c^2-2bc\cos(B-C)$, are in G. P.

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161. Solve $\cos 3\theta+\sin 3\theta=\cos\theta+\sin\theta$.

162. If $x=3\cos\phi+\cos 3\phi$, $y=3\sin\phi-\sin 3\phi$, then
$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=4^{\frac{2}{3}}.$$

163. $$\cos^2 18^0\sin^2 36^0-\cos 36^0\sin 18^0=\tfrac{1}{16}.$$

164. Given that $\cos\theta=-\frac{1}{2}$ is one solution of the equation $\cos\theta+\cos 3\theta=\frac{1}{2}$, find the others.

165. If the diagonal $BD$ of a quadrilateral inscribed in a circle passes through the centre its area $=(s-a)(s-d)=(s-b)(s-c)$.

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166. If $$\tan(A+B)=3\tan A,$$
then $$\sin(2A+2B)+\sin 2A=2\sin 2B.$$

167. $\sin 18^0+\cos 18^0=\sqrt{2}\cos 27^0$.

168. Solve $\tan 3\theta+\tan 2\theta+\tan\theta=0$.

169. If $A+B+C=90^0$, then $\cos^3 2A+\cos^3 2B+\cos^3 2C$
$=3\sin A\sin B\sin C-\sin 3A\sin 3B\sin 3C+1$.

170. In any triangle $bc\cos^2\frac{1}{2}A+ca\cos^2\frac{1}{2}B+ab\cos^2\frac{1}{2}C=(s)^2$.

---

171. Prove that $7\theta=\pi$ satisfies the equation
$$8\cos\theta\cos 2\theta\cos 3\theta=1.$$

172. If $A+B+C=90^0$, then $\cos A\cos B\cos C$ cannot be greater than $\frac{3}{8}\sqrt{3}$.

173. $$\sin^{-1}\tfrac{1}{5}\sqrt{5}+\cot^{-1}3=\tfrac{1}{4}\pi.$$

174. In any triangle
$$\sin 3A+\sin 3B+\sin 3C+4\cos\tfrac{3}{2}A\cos\tfrac{3}{2}B\cos\tfrac{3}{2}C=0.$$

175. If the circle circumscribing an isosceles triangle is equal to the escribed circle touching one of the equal sides the triangle is right angled.

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176. If $\sin x+\sin^2 x=1$, then $\cos^2 x+\cos^4 x=1$.

177. $\sin 10^0 \sin 50^0 \sin 70^0=\frac{1}{8}$.

178. Simplify $\tan^{-1}\left(\dfrac{x\cos\theta}{1-x\sin\theta}\right)-\cot^{-1}\left(\dfrac{\cos\theta}{x-\sin\theta}\right)$.

179. $A$, $B$, $C$, $D$ are the angles of a quadrilateral; prove that the sum of the products of the sines of $\frac{1}{2}A$, $\frac{1}{2}B$, $\frac{1}{2}C$, $\frac{1}{2}D$ taken two together is equal to the sum of the products of their cosines taken two together.

180. At each end of a horizontal line of length $2a$ the altitude of a certain tower is $\alpha$ and at the middle point of the line it is $\beta$; show that the vertical height of the tower above the horizontal plane is

$$a\sin\alpha\sin\beta\sqrt{\{\operatorname{cosec}(\beta+\alpha)\operatorname{cosec}(\beta-\alpha)\}}.$$

---

181. Prove that

$$\sin(\alpha-\beta)\cos 2\beta+\cos(\alpha-\beta)\sin 2\beta=\sin(\beta-\alpha)\cos 2\alpha+\cos(\beta-\alpha)\sin 2\alpha.$$

182. Find the values of $\sin 2\theta \cos \frac{1}{2}\theta$ and $\tan\theta$ when $\theta=\sin^{-1}\frac{5}{13}$.

183. Prove geometrically that $\sin 2\theta$ and $2\sin\theta\cos\theta$ are always of the same sign.

184. If $D$ is the middle point of the side $BC$, then

$$4AD^2=b^2+c^2+2bc\cos A.$$

185. If $l$, $m$, $n$ are the distances of the centre of the inscribed circle from $ABC$, then

$$\frac{lmn}{r}=\frac{2abc}{a+b+c}.$$

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186. If $a\sin\theta=b\sin\phi$, $c=a\cos\theta+b\cos\phi$, then

$$\tan(\theta+\phi)(c^2-a^2-b^2)=\sqrt{\{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}.$$

187. If $\sin^{-1}\alpha+\sin^{-1}\beta+\sin^{-1}\gamma=\pi$, then

$$\alpha\sqrt{(1-\alpha^2)}+\beta\sqrt{(1-\beta^2)}+\gamma\sqrt{(1-\gamma^2)}=2\alpha\beta\gamma.$$

188.
$$\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}+\frac{\sin(\theta-\gamma)\sin(\theta-\alpha)}{\sin(\beta-\gamma)\sin(\beta-\alpha)}+\frac{\sin(\theta-\alpha)\sin(\theta-\beta)}{\sin(\gamma-\alpha)\sin(\gamma-\beta)}=1.$$

189. If $A+B+C=90$, then

$\sin^2 A+\sin^2 B+\sin^2 C$ cannot be less than $\frac{3}{4}$.

190. The perimeter of a regular circumscribing polygon of a circle is double that of the inscribed polygon having the same number of sides; what is that number?

191. Eliminate $\theta$ from the equations

$$m \sin 2\theta = n \sin \theta, \quad p \cos 2\theta = q \cos \theta.$$

192. Reduce $1 - \cos^2 \theta - \cos^2 \phi - \cos^2 \psi + 2 \cos \theta \cos \phi \cos \psi$ to the product of four cosines.

193. $$\tan 5A = \frac{5 \tan A - 10 \tan^3 A + \tan^5 A}{1 - 10 \tan^2 A + 5 \tan^4 A}.$$

194. Prove (i) $4 \sin 18^0 \cos 36^0 = 1$,

(ii) $\cos 36^0 - \sin 18^0 = \frac{1}{2}$.

195. $r = s \tan \frac{1}{2}A \tan \frac{1}{2}B \tan \frac{1}{2}C$.

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196. Solve the equation

$$\cos \theta - \sin \theta = \cos \alpha - \sin \alpha.$$

197. Prove that when

$$\cos (x + 2A) + \cos (x + 2B) + \cos (x + 2C) + \cos (x + 2D) = 4 \cos (A + B) \cos (A + C) \cos (A + D),$$

then $$\sin (x + 2A) + \sin (x + 2B) + \sin (x + 2C) + \sin (x + 2D) = 4 \sin (A + B) \sin (A + C) \sin (A + D),$$

where $A$, $B$, $C$, $D$ are the angles of a quadrilateral and $x$ is not zero.

198. Express in four factors

$$\sin^2 A + \sin^2 B + \sin^2 C - 2 \sin A \sin B \sin C - 1.$$

199. In any triangle

$$2R \{\cos^2 \tfrac{1}{2}A + \cos^2 \tfrac{1}{2}B + \cos^2 \tfrac{1}{2}C\} = 4R + r.$$

200. The diagonals of a four-sided figure are $h$ and $k$, and the area is $C$; show that the area of its circumscribing square is

$$\frac{h^2k^2 - 4C^2}{h^2 + k^2 - 4C}.$$

## * * MISCELLANEOUS EXAMPLES. LXXVI.

(1) If $2\cos\theta - \cos 2\theta = a$ and $2\sin\theta - \sin 2\theta = b$, prove that $(a^2+b^2-3)^2 = 12-8a$.

(2) If $h\cos\theta + k\sin\theta = 1$ and $l\cos\theta + m\sin\theta = 1$, prove that $(l-h)^2+(m-k)^2=(lh-mk)^2$.

(3) The diagonals of a rhombus are $2a$ and $2b$, prove that the cosines of its angles are $\pm\dfrac{a^2 \sim b^2}{a^2+b^2}$.

(4) One of the values of

$\sqrt{}[2+\sqrt{}\{2+\sqrt{}(2+\ldots+\sqrt{2+2\cos A})\}]$ is $2\cos\dfrac{A}{2^n}$.

(5) If $8x = \log_e 3$, prove that the angle whose tangent is $\dfrac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$ is $15^0$.

(6) If the number of degrees ($A$) in an angle is the same as the number of radians in another angle, and if the tangents of these angles are equal, prove that $A$ is some multiple of $\dfrac{180\pi}{180-\pi}$.

(7) If $\alpha$, $\beta$, $\gamma$ are in A.P., then

$$\sin\alpha + \sin\gamma = 2\sin\beta \,.\, \cos(\beta-\alpha).$$

(8) Two parallel chords of a circle, lying on the same side of the centre, subtend respectively $72^0$ and $144^0$ at the centre. Prove that the distance between the chords is half the radius of the circle.

(9) Prove that $4\sin(\theta-\alpha) \,.\, \sin(m\theta-\alpha) \,.\, \cos(\theta-m\theta)$

$$= 1+\cos(2\theta-2m\theta)-\cos(2\theta-2\alpha)-\cos(2m\theta-2\alpha).$$

(10) Express $x^4+y^4+z^4-2y^2z^2-2z^2x^2-2x^2y^2$ in a form fitted for logarithmic calculation.

(11) Solve the equations

(i) $\sin 3\theta + \sqrt{3}\cos 3\theta = 1.$ (ii) $\sin m\theta = \cos n\theta.$

(iii) $\dfrac{\cos(\beta + x)}{\cos(\alpha - x)} = \dfrac{m \sin\beta}{n \sin\alpha}.$ (iv) $\tan m\theta = \cot n\theta.$

(v) $\tan\theta + \tan 2\theta + \tan 3\theta = 0.$

(vi) $\cos 8\theta - \cos 5\theta + \cos 3\theta = 1.$

(vii) $\cos\theta . \cos 3\theta = \cos 5\theta . \cos 7\theta.$

(12) In any triangle $\tan\dfrac{A-B}{2} = \tan\left(\phi - \dfrac{\pi}{4}\right)\cot\dfrac{C}{2}$, where $\phi = \tan^{-1}\dfrac{a}{b}$.

(13) Prove that

$$b^2 + c^2 - 2bc\cos(60^\circ + A) = c^2 + a^2 - 2ca\cos(60^\circ + B).$$

Hence prove that if equilateral triangles are described on the sides of the triangle $ABC$, the centres of circles inscribed in them will be the angular points of an equilateral triangle.

(14) A round tower stands on an island in a lake. $A$, $B$ are two points on the land such that $AB$ is $a$ feet and points directly to the middle of the tower. At $A$ and $B$ the base of the tower subtends the angles $2\alpha$ and $2\beta$ respectively. Prove that the diameter of the tower is $2\dfrac{a\sin\alpha . \sin\beta .}{\sin\beta - \sin\alpha}$.

(15) At $P$ the top of a tower of height $h$, the angles of depression of two objects on the horizontal plane on which the tower stands are $\frac{1}{4}\pi - \alpha$ and $\frac{1}{4}\pi + \alpha$. Prove that the angle $APB = 2\alpha$, and that $AB = 2h\tan 2\alpha$.

(16) On the side of a hill there are two places $B$, $C$ inaccessible to each other, but known to be at the same distance ($a$) from a certain station $A$ also on the hill. At the lower place $C$ the *horizontal* angle $\theta$ between $A$ and $B$ is observed as well as the altitudes $\lambda$, $\mu$ of $A$ and $B$. Prove that the distance between $B$ and $C$ is $2a\left\{\cos(\lambda - \mu)\cos^2\dfrac{\theta}{2} - \cos(\lambda + \mu)\sin^2\dfrac{\theta}{2}\right\}$.

(17) In the ambiguous case, given $B$, $c$, $b$, if $a_1$, $a_2$ are the two values of $a$, prove the following statements:

(i) $a_1+a_2=2c\cos B$. (ii) $a_1a_2=c^2-b^2$.

(iii) $a_1^2-2a_1a_2\cos 2B+a_2^2=4b^2\cos^2 B$.

(iv) The distance between the centres of the circumscribing circles of the two triangles is $\frac{a_1-a_2}{2\sin B}$.

(v) The circumscribing circles of the two triangles are equal.

(vi) If $B=45^\circ$, the angle between the two positions of $b$ is

$$\cos^{-1}\frac{2a_1a_2}{a_1^2+a_2^2}.$$

(18) Prove that if $I_1$, $I_2$, $I_3$ are the centres of the escribed circles, then $I_2AI_3$ is a straight line, and $I_1A$, $I_2B$, $I_3C$ are perpendicular to $I_2I_3$, $I_3I_1$, $I_1I_2$ respectively. Prove also that

(i) $I_2I_3=a\operatorname{cosec}\frac{A}{2}$. (ii) The angle $I_2I_1I_3=\frac{\pi}{2}-\frac{A}{2}$.

(iii) The area of $I_1I_2I_3=\frac{sa}{\sin A}=\frac{\frac{1}{4}S}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$.

(iv) The radius of the circle circumscribing $I_1I_2I_3=2R$.

(19) If $D$, $E$, $F$ are the feet of the perpendiculars from $A$, $B$, $C$ on the opposite sides, the triangle $DEF$ is called the 'pedal triangle' of the triangle $ABC$*; prove that

(i) $EF=a\cos A$. (ii) The angle $EDF=\pi-2A$.

(iii) $\frac{1}{r}=\frac{1}{AD}+\frac{1}{BE}+\frac{1}{CF}$. (iv) $\frac{AD^2}{BE.CF}=\frac{bc}{a^2}$.

(v) The radius of the circle circumscribing $AEF=R\cos A$.

(vi) The radius of the circle circumscribing $DEF=\frac{1}{2}R$.

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* The student should notice that $ABC$ is the pedal triangle of $I_1I_2I_3$ in Question (18).

(20) Prove that in any triangle $ABC$

$$a^2 = (b-c)^2 \left\{1 + \frac{4bc}{(b-c)^2} \sin^2 \frac{A}{2}\right\}.$$

Hence prove that if $\tan \theta = \frac{2\sqrt{bc}}{b-c} \sin \frac{A}{2}$, then $a = (b-c) \sec \theta$.

(21) Prove that in any triangle $ABC$

$$a = (b+c) \cos \beta, \text{ where } \sin \beta = \frac{2\sqrt{bc}}{b+c} \cos \frac{A}{2}.$$

Use this method, or that of Example (20), to find $a$ by the aid of the Tables, when $b = 347$, $c = 293$, $A = 39^0\ 42'$. [See Art. 260.]

(22) Prove that in any triangle

$$\log a = \log (b-c) + L \cos \frac{A}{2} - L \cos \phi\,; \text{ where } \phi = \tan^{-1} \frac{b+c}{b-c} \tan \frac{A}{2}.$$

(23) If $D$, $E$, $F$ are the middle points of $BC$, $CA$, $AB$, then

$$\sin BAD = \frac{b \sin A}{\sqrt{b^2 + 2bc \cos A + c^2}}.$$ [See Ex (11) (15) page 205.]

(24) If $x = II_1$, $y = II_2$, $z = II_3$, $d = 2R$, prove that

$$xyz + d(x^2 + y^2 + z^2) = 4d^3.$$

(25) The circle escribed to the side $BC$ of the triangle $ABC$ will be the inscribed circle of the triangle whose sides are

$$\frac{sa}{s-a}, \frac{sb}{s-a}, \frac{sc}{s-a}.$$

(26) If a point $Q$ be taken within $ABC$ such that the angles $BQC$, $CQA$, $AQB$ are each $120^0$, prove that

$$OA = \frac{1}{2} \cdot \frac{4\sqrt{2}\Delta + \sqrt{6}(b^2 + c^2 - a^2)}{\{12\sqrt{3}\Delta + 3(a^2 + b^2 + c^2)\}^{\frac{1}{2}}}.$$

(27) If $\sin A$ and $\cos A$ are both given, prove that in general $n$ different values and no more can be assigned to $\sin \frac{A}{n}$.

(28) If from a point on a circular arc (of radius $r$) which subtends an angle $a$ at the centre, perpendiculars are drawn to its bounding radii, then the distance between the feet of the perpendiculars is $r \sin a$.

(29) If two circles, radii $a$ and $b$, touch externally, the angle between their common tangents $= 2 \tan^{-1} \frac{1}{2} \left( \sqrt{\frac{a}{b}} \sim \sqrt{\frac{b}{a}} \right)$.

(30) If the inscribed circle touches the sides in $DEF$, prove that $EF : FD : DE = \cos \frac{A}{2} : \cos \frac{B}{2} : \cos \frac{C}{2}$.

(31) If the bisectors of the angles $A$, $B$, $C$ meet the opposite sides in $D, E, F$ then (i) $AD = \frac{2bc}{b+c} \cos \frac{A}{2}$. (ii) $CD = \frac{a \sin B}{\sin B + \sin C}$.

(iii) The area of $DEF$

$$= S \frac{2abc}{(b+c)(c+a)(a+b)} = S \cdot \frac{2 \sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2}}{\cos \frac{B-C}{2} \cdot \cos \frac{C-A}{2} \cdot \cos \frac{A-B}{2}}.$$

(32) If $a, \beta, \gamma, \delta$ are the angles of a quadrilateral inscribed in a circle, then

$$\cos(a+\beta) \cdot \cos(\beta+\gamma) \cdot \cos(\gamma+\delta) \cdot \cos(\delta+a) = (1 - \cos^2 a - \cos^2 \beta)^2.$$

(33) If $\theta_1$ and $\theta_2$ be two values of $\theta$ found from the equation $a \cos \theta + b \sin \theta = c$, then

$$\frac{1}{a} \cdot \cos \frac{\theta_1 + \theta_2}{2} = \frac{1}{b} \cdot \sin \frac{\theta_1 + \theta_2}{2} = \frac{1}{c} \cdot \cos \frac{\theta_1 - \theta_2}{2}.$$

(34) If $\left.\begin{aligned} a &= b \cos \chi + c \cos \phi \\ b &= c \cos \theta + a \cos \chi \\ c &= a \cos \phi + b \cos \theta \end{aligned}\right\}$

prove that $\cos 2\theta + \cos 2\phi + \cos 2\chi + 4 \cos \theta \cdot \cos \phi \cdot \cos \chi + 1 = 0$, hence [see Ex. (20), p. 143], prove that

$$(\theta \pm \phi \pm \chi) = (2n+1)\pi.$$

(35) A circle is drawn touching the inscribed circle and the sides $AB$, $AC$ (not produced) of the triangle $ABC$. Another circle is drawn touching the circle and the same side. Prove that the radius of the $n^{th}$ circle thus drawn is $r\left(\dfrac{1-\sin\frac{A}{2}}{1+\sin\frac{A}{2}}\right)^n$ and that the sum of the radii of all the circles, when $n=\infty$, is

$$\frac{r}{2}\left(\operatorname{cosec}\frac{A}{2}-1\right).$$

(36) If $AO$, $BO$, $CO$ pass through the centre of the circumscribing circle and meet the opposite sides on $DEF$, then

$$\frac{1}{AD}+\frac{1}{BE}+\frac{1}{CF}=\frac{2}{R}.$$

(37) If $h$ and $k$ be the lengths of the diagonals of a quadrilateral and $\theta$ the angle between them, prove that the area of the quadrilateral is $\frac{1}{2}hk\sin\theta$.

(38) If $\dfrac{x}{\sin X}=\dfrac{y}{\sin Y}=\dfrac{z}{\sin Z}$ and $X+Y+Z=180^0$, prove that $x=y\cos Z+z\cos Y$.

(39) Two regular polygons of $n$ and $2n$ sides are described such that the circle inscribed in the first circumscribes the second; also the radius of the circle inscribed in the second is to that of the circle circumscribing the first as $3+\sqrt{3}$ is to $4\sqrt{2}$. Prove that $n=6$.

(40) Prove by the aid of a figure like that of Art. 282 that $I_1O^2=R^2+2Rr_1$.

(41) If $yz+zx+xy=1$, prove by Trigonometry that

$$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}.$$

[See Ex. (33), (32), p. 194.]

# EXAMINATION PAPERS.

## †I. PREVIOUS EXAMINATION.

*December*, 1886.

### PAPER I.

I. Define the secant and cotangent of an angle, and prove that

(i) $\sec\theta\,\mathrm{cosec}\,\theta = \tan\theta + \cot\theta$,

(ii) $\cot^2\theta\,\dfrac{\sec\theta - 1}{1 + \sin\theta} + \sec^2\theta\,\dfrac{\sin\theta - 1}{1 + \sec\theta} = 0$.

### PAPER II.

1. From the definitions of the Trigonometrical Functions, prove that

$$\sin^2 A = 1 - \cos^2 A \quad \text{and} \quad \sin A \tan A = \sec A - \cos A.$$

Prove that $\frac{1}{3}(\cos^6 A + \sin^6 A) - \frac{1}{4}(\cos^2 A - \sin^2 A)^2 = \frac{1}{12}$.

2. Investigate the values of $\tan 45^0$ and $\sin 60^0$.

Two adjacent sides of a parallelogram are of lengths 15 and 24, and the angle between them is $60^0$; find the lengths of both diagonals.

## II. PREVIOUS EXAMINATION, CAMBRIDGE.

*June*, 1887.

### PAPER I.

1. Define the sine and tangent of an angle, and shew how to find the sine and tangent whose cosine ($m$) is given.

If $\sin A = \tan B$, prove

$$\cos^2 A \cos^2 B = (\cos B + \sin B)(\cos B - \sin B).$$

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† The Additional Subjects in the Cambridge Previous Examination, which are required of Candidates for any Tripos, are now (1887) *either* Mathematics, *or* French, *or* German.

The Mathematical subjects are (i) The Trigonometry of one Angle, (ii) Elementary Dynamics [see Lock's *Dynamics for Beginners*], (iii) Elementary Statics. The Trigonometry is set in the first one or two questions in each of the two papers. The questions quoted above were set in December, 1886, and in June, 1887.

2. Trace the changes in the tangent of an angle as the angle changes from $180^0$ to $270^0$.

If $\sin\theta = -\frac{2}{3}$, find $\tan\theta$; and explain by means of a figure the reason why there are two answers to the question.

### PAPER II.

1. Explain the mode of measuring angles in degrees, minutes and seconds.

Find the number of seconds of angle through which the earth revolves about its axis in a second of time.

2. Find the value of $\sec 60^0$, and of $\sec 45^0$.

Solve the equation

$$2\operatorname{cosec} A = 2\sin A + \cot A.$$

## III. CAMBRIDGE LOCAL EXAMINATIONS.

### *Dec.* 1886.

### JUNIOR CANDIDATES.

1. Prove that the angle subtended at the centre of a circle by an arc equal to the radius is the same for all circles.

Express the angle as a fraction of a right angle.

2. Define the sine, cosine and tangent of an angle. Prove that these trigonometrical ratios are always the same for the same angle.

Find these ratios for an angle of $45^0$.

3. Prove that

$$\cos(A+B) = \cos A\cos B - \sin A\sin B.$$

Prove that the sum of the cosines of two angles is equal to twice the cosine of half their sum multiplied by the cosine of half their difference.

4. Prove the following relations:

(i) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$.

(ii) $(1 + \sin A + \cos A)^2 = 2(1 + \sin A)(1 + \cos A)$.

(iii) $\dfrac{\sin 3A + \sin 5A}{\cos 3A - \cos 5A} = \cot A$.

5. Prove that the logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers.

Having given $\log 2 = \cdot 3010300$, $\log 7 = \cdot 8450980$, find the logarithms of $(1\cdot 75)^{\frac{1}{2}}$, $(24\cdot 5)^{-\frac{1}{3}}$.

6. Prove that in any triangle $ABC$

(i) $a = b\cos C + c\cos B$.

(ii) $2bc\cos A = b^2 + c^2 - a^2$.

7. Shew how to solve a triangle when one side and two angles are known.

Find the side $b$ in the triangle $ABC$ from the following data:

$$a = 156\cdot22,\quad B = 57^\circ25',\quad C = 63^\circ42',$$
$$\log 15\cdot622 = 1\cdot1937366,\quad L\sin 57^\circ25' = 9\cdot9256261,$$
$$\log 15\cdot37552 = 1\cdot1868297,\quad L\sin 58^\circ53' = 9\cdot9325330.$$

8. The angles of elevation of the top of a tower on a horizontal plane observed at two points distant $a$ feet and $b$ feet respectively from the base and in the same straight line with it are found to be complementary. Shew that the height of the tower is $\sqrt{ab}$ feet. If $\theta$ be the angle subtended at the top of the tower by the line joining the two points, prove that

$$\sin\theta = \frac{a \sim b}{a+b}.$$

## IV. OXFORD LOCAL EXAMINATIONS.

### *July*, 1887.

### Junior Candidates.

1. What is $\pi$?

What is 'the angle whose circular measure is $\pi$'?

In a triangle $ABC$ the angle $A$ is $x$ degrees, the angle $B$ $x$ grades, and the circular measure of $C$ is $\frac{\pi x}{9}$; find the number of degrees in each of the angles.

2. Prove that

$$2\sin\frac{A}{2} = \pm\sqrt{1+\sin A} \pm \sqrt{1-\sin A},$$

and determine which are the correct signs when

$$270^\circ > A > 180^\circ.$$

3. Obtain the following formulæ:

(1) $\cos(A+B) = \cos A\cos B - \sin A\sin B$;

(2) $\tan A + \tan B = \sin(A+B)\sec A\sec B$;

(3) $\dfrac{\sin(p+q) - 2\sin p + \sin(p-q)}{\cos(p+q) - 2\cos p + \cos(p-q)} = \tan p$;

(4) $\sin 2\theta \sin 2\phi = \sin^2(\theta+\phi) - \sin^2(\theta-\phi)$.

4. Given $\log 2 = \cdot 3010300$,

$\log 3 = \cdot 4771213$, $\log 24668 = 4\cdot 3921339$,

$\log 11 = 1\cdot 0413927$, $\log 24669 = 4\cdot 3921515$;

find the logarithm of 30·25 and calculate the value of

$$\{165 \times (30)^9 \times \sqrt[3]{24}\} \div (121)^7.$$

5. In a triangle $ABC$, $a$, $b$, $c$ are the sides, $s$ is the semi-perimeter, $\Delta$ the area, $R$, $r$ the radii of the circumscribed and inscribed circles: prove that

(1) $\tan \dfrac{B-C}{2} = \dfrac{b-c}{b+c} \cot \dfrac{A}{2}$;

(2) $\sin \dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$;

(3) $2R = a \operatorname{cosec} A = b \operatorname{cosec} B = c \operatorname{cosec} C$;

(4) $2\Delta = ca \sin B$.

6. In a plane triangle the sides $a$, $b$ and the angle $A$ are known; shew that in general two values of $c$ can be found, and that the difference of these values is

$$2\sqrt{a^2 - b^2 \sin^2 A}.$$

7. A ladder placed at an angle of 75° just reaches the sill of a window 27 feet above the ground on one side of a street. On turning the ladder over without moving its foot, it is found that, when it rests against a wall on the other side of the street, it is at an angle of 15°. Find the breadth of the street.

8. If $B = 36^\circ 46'$, $b = 311\cdot 8785$, $c = 521\cdot 05$, find $C$.

[$L \sin B = 9\cdot 7771060$, $\log 31187 = 4\cdot 4939736$,

$\log 521\cdot 05 = 2\cdot 7168794$, $\log 31188 = 4\cdot 4939875$.]

## V. CAMBRIDGE LOCAL EXAMINATIONS.

### *December*, 1886.

### SENIOR STUDENTS.

1. Explain the method of measuring angles in circular measure, and find the circular measure of the angle of a regular pentagon.

Prove that

$$\frac{\tan^3 A}{1+\tan^2 A} + \frac{\cot^3 A}{1+\cot^2 A} = \frac{1-2\sin^2 A \cos^2 A}{\sin A \cos A}.$$

2. Establish the identities:

(1) $\tan^4 A = \dfrac{3 - 4\cos 2A + \cos 4A}{3 + 4\cos 2A + \cos 4A}$.

(2) $\sin A - \sin B = 2 \sin \dfrac{A-B}{2} \cos \dfrac{A+B}{2}$.

If $A+B+C=180^\circ$, shew that

$$(\sin B-\sin C)\cot\frac{A}{2}+(\sin C-\sin A)\cot\frac{B}{2}+(\sin A-\sin B)\cot\frac{C}{2}=0.$$

3. Find an expression for all the angles which have a given sine. Solve the equations:

(1) $\cos\theta+\tan\theta=\sec\theta$.

(2) $\sin\theta-2\sin 2\theta\cos\theta+\cos 3\theta=\cos 2\theta$.

4. Prove that in any triangle if $a$, $b$, $c$ be the sides opposite to the angles $A$, $B$, $C$

(1) $\dfrac{\sin A}{a}=\dfrac{\sin B}{b}=\dfrac{\sin C}{c}$,

(2) $\cos A+\cos B+\cos C=1+\dfrac{r}{R}$,

where $r$ and $R$ are the radii of the inscribed and circumscribed circles respectively.

If the line joining $A$ to the centre of the inscribed circle meets the opposite side in $D$, prove that

$$\tan ADB=\frac{b+c}{b-c}\tan\frac{A}{2}.$$

*5. If $\theta$ is the circular measure of a positive angle less than a right angle, shew that $\sin\theta$ lies between $\theta$ and $\theta-\dfrac{\theta^3}{6}$.

Find the limiting value of $\left(\dfrac{n}{\theta}\sin\dfrac{\theta}{n}\right)^n$ when $n$ is indefinitely increased.

*6. Prove that, if $\theta$ lie between $\dfrac{\pi}{4}$ and $-\dfrac{\pi}{4}$,

$$\theta=\tan\theta-\tfrac{1}{3}\tan^3\theta+\tfrac{1}{5}\tan^5\theta-\ldots\ldots.$$

Shew that

$$\log(a+b\sqrt{-1})=\tfrac{1}{2}\log(a^2+b^2)+\sqrt{-1}\tan^{-1}\frac{b}{a}.$$

* See *Higher Trigonometry*.

## VI. OXFORD LOCAL EXAMINATIONS.

### *July*, 1887.

### SENIOR CANDIDATES.

1. Prove geometrically that the sine of $90^\circ+A$ is equal to the cosine of $A$ for all values of $A$.

Find all the values of $B$, less than $180^\circ$, for which $\sin 5B=\tfrac{1}{2}\sqrt{2}$.

2. (1) Prove that $2\sin A = \pm\sqrt{1+\sin 2A} \pm \sqrt{1-\sin 2A}$; and determine the signs of the roots when $2A$ is greater than 180° and less than 270°.

(2) Verify the identities:

($\alpha$) $\sin 4A = 4\cos A\,(\sin A - 2\sin^3 A)$;

($\beta$) $(\tan 4A + \tan 2A)(1 - \tan^2 3A\,\tan^2 A) = 2\tan 3A\,\sec^2 A$.

3. Prove the following properties of a triangle $ABC$, of which the sides are $a$, $b$, $c$, and the semi-perimeter is $s$:

(1) area $= \sqrt{s(s-a)(s-b)(s-c)}$;

(2) $$\frac{bc\cos A + ca\cos B + ab\cos C}{bc+ca+ab} = \frac{a\sin A + b\sin B + c\sin C}{(b+c)\sin A + (c+a)\sin B + (a+b)\sin C}.$$

4. $ABC$ is an isosceles triangle, having the vertical angle $A = 56°\,30'$; on $AB$ as base an isosceles triangle $DAB$ is described, having its vertical angle $D = 38°\,40'$: the perpendicular from $A$ upon $BC = 22{\cdot}75$ inches: find the length of the perpendicular from $D$ upon $AB$ to four places of decimals.

$L\cot 19°\,20' = 10{\cdot}4548807$; $L\cos 28°\,15' = 9{\cdot}9449220$;
$\log 11375 = 4{\cdot}0553514$;
$\log 36754 = 4{\cdot}5653046$, diff. for $1 = 119$.

5. (1) If in a triangle the radius of the circumscribed circle is double of the radius of the inscribed circle, the triangle is equilateral.

(2) Compare the areas of two equilateral and equiangular polygons, each of $n$ sides, one inscribed in and the other circumscribed about the same circle.

## VII. WOOLWICH.

*December*, 1886.

[N.B.—*Great importance will be attached to accuracy.*]

1. Explain what is meant by the *circular measure* of an angle, and shew that the circular measure of an acute angle is intermediate in value between the sine and the tangent of the angle.

The perimeter of a certain sector of a circle is equal to the length of the arc of a semicircle having the same radius. Express the angle of the sector in degrees, minutes, and seconds.

2. Prove that the secant of any angle will be either greater than $+1$ or less than $-1$.

Shew that $\sec A = \pm\sqrt{1+\tan^2 A}$, and explain the appearance of the double sign.

3. Express $\sin(A-B)$ and $\cos(A-B)$ in terms of the sines and cosines of $A$ and $B$.

Find the values of sin 15°, sin 105°, cos 165°, and tan 195°.

4. Obtain a formula including all angles which have a given tangent.

If $\tan A + \tan 2A = \tan 3A$, prove that $A$ must be a multiple of $60^\circ$ or $90^\circ$.

5. A ring, 10 inches in diameter, is suspended from a point 1 foot above its centre by six equal strings attached to its circumference at equal intervals. Find the cosine of the angle between two consecutive strings.

6. Prove that

(1) $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$.

(2) $\cos^2 A - \cos A \cos(60^0 + A) + \sin^2(30 - A) = \frac{3}{4}$.

7. State and prove the rule for finding, by inspection, the characteristic of the logarithm of a fraction.

Find from the tables supplied an approximate value of the seventh root of ·000026751.

8. Express the cosine of any angle of a triangle in terms of the three sides.

Prove that

$$\sin(B - C) : \sin(B + C) :: b^2 - c^2 : a^2.$$

9. The sides of a triangle are 17, 20, and 27. Find from the tables supplied all the angles.

10. A man travelling due west along a straight road observes that when he is due south of a certain windmill the straight line drawn to a distant church makes an angle of $30^0$ with the direction of the road. A mile further on the bearings of the windmill and tower are N.E. and N.W. respectively. Find the distances of the tower from the windmill, and from the nearest point of the road.

11. Find the area of a regular polygon of $n$ sides inscribed in a circle of given radius.

If a regular pentagon and a regular decagon have the same perimeter, prove that their areas are as $2 : \sqrt{5}$.

## VIII. SANDHURST FURTHER.

*December*, 1885.

1. Define the two common units of angular measure.

Find the circular measure of $42^0$, and find the angle whose circular measure is $\frac{5}{6}$.

2. Assuming that

$$\left.\begin{aligned}\sin(A+B) &= \sin A\cos B + \cos A \sin B\\ \cos(A+B) &= \cos A \cos B - \sin A \sin B\end{aligned}\right\},$$

find in terms of the ratios of $A$, the values of

$$\sin 2A,\quad \cos 2A,\quad \tan 2A,\quad \sin \tfrac{1}{2}A,\quad \cos\tfrac{1}{2}A \text{ and } \tan\tfrac{1}{2}A.$$

3. Prove that

$$1-\tan^2 A\tan^2 B=\frac{\cos^2 B-\sin^2 A}{\cos^2 A\cos^2 B};$$

and from the equation $2\tan^2\theta=\sec^2\theta$ find a general expression for $\theta$.

4. Prove that in any triangle

$$\cos A+\cos B=\frac{2(a+b)}{c}\sin^2\tfrac{1}{2}C.$$

A lighthouse appears to a man in a boat 300 yards from its foot, to subtend an angle of 6° 20′ 27·7″; find in feet the height of the lighthouse, having given

$L\tan 6^\circ 20'=9{\cdot}0452836$, difference for $1={\cdot}0011507$; $\log 3={\cdot}4771213$.

5. Prove that in any triangle

$$\tan 2A+\tan 2B+\tan 2C=\tan 2A\tan 2B\tan 2C.$$

Hence shew that if $x$, $y$, $z$ are three numbers such that

$$x+y+z=xyz,$$

then $x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y)=4xyz.$

## IX. SANDHURST FURTHER.

*June*, 1886.

1. Express the value of the tangent, secant, and cosecant in terms of the sine of the angle, and also in terms of the cosine of twice the angle.

Find the values of the tangent, secant, and cosecant of 22° 30′.

2. Prove the following:

$$(\sin 2A)^2=2\cos^2 A(1-\cos 2A),$$

$$\tan 6A=\frac{\cos 5A-\cos 7A}{\sin 7A-\sin 5A};$$

and if $\frac{a}{b}=\frac{\cos A}{\cos B}$,

prove that $a\tan A+b\tan B=(a+b)\tan\tfrac{1}{2}(A+B)$.

3. Shew that in any triangle

$$(b+c-a)\tan\tfrac{1}{2}A=(c+a-b)\tan\tfrac{1}{2}B=(a+b-c)\tan\tfrac{1}{2}C$$
$$=\left\{\frac{(b+c-a)(c+a-b)(a+b-c)}{a+b+c}\right\}^{\frac{1}{2}}.$$

If $R$, $r_a$, $r_b$, $r_c$ are the radii respectively of the circumscribed and three escribed circles of a triangle, shew that

$$Rr_a(s-a)=Rr_b(s-b)=Rr_c(s-c)=\tfrac{1}{4}abc,$$

where $S$ is the semi-perimeter of the triangle.

4. Find the area of a triangle in terms of (i) two sides and the angle between them, (ii) two angles and the side between them.

5. Define the characteristic and the mantissa of a logarithm. Find the logarithm of 5 when the base is 3, and the logarithm of $\frac{1}{3}$ when the base is 5;

$$\log_{10} 3 = \cdot 4771213, \quad \log_{10} 2 = \cdot 3010300.$$

6. Two sides of a triangle are 9 and 7, and the included angle is $38^\circ\, 56'\, 32\cdot 8''$; find the base and remaining sides

$$L \tan 19^\circ 29' = 9\cdot 5487471, \quad L \tan 19^\circ\, 28' = 9\cdot 5483452.$$

## X. COLLEGE OF PRECEPTORS, PUPILS' EXAMINATION.

*July*, 1886.

1. In a quadrilateral $ABCD$, the angle $A = 30^\circ$, $B = 60$ grades, $C = \frac{2}{3}\pi$; find the number of degrees in $D$. Find what number of degrees must be taken from the angle $D$ and added to $A$ so that the figure may be inscribed in a circle.

2. If $\sec A = m$, what will $\cot A$ be? Shew from your result that when $A = 0$ or $180^\circ$ the cotangent will be infinite in both cases.

3. Prove $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

Shew that if $A + B + C = 180^\circ$,

$$\sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C;$$

and that this is true numerically if $A$, $B$, $C$ the angle of a triangle be 30, 60, 90 degrees respectively.

4. Solve $\cot^2 A + \tan^2 A = \frac{10}{3}$; giving the general value of $A$.

5. If $\log_a 243 = 5$, find $a$.

Given $\log_{10} 30 = 1\cdot 4771213$, $\log_{10} 20 = 1\cdot 3010300$, find $\log_{10} 72$ and $\log_{10} 1\cdot 44$.

6. In any triangle, prove the formula

$$\tan \frac{A}{2} = \pm \sqrt{\left\{ \frac{(s-b)(s-c)}{s(s-a)} \right\}},$$

and shew from it that if $a = 5$, $b = 3$, $c = 4$, then $A$ will be a right angle. Can you give any explanation of the double sign?

7. Find the area of a regular polygon of $n$ sides, each side being $a$.

8. Two angles of a triangle $A$ and $B$, and their included side $C$ being given, find an expression for the area; shew from the result that if $A$ and $B$ each remain constant while the side $C$ varies, Euclid's statement is true; 'that the areas of similar rectilineal figures vary in the duplicate ratio of their homologous sides.'

9. Standing at a certain point, I observe the elevation of a house to be $15^\circ\,15'$, and that the sill of one of its windows, known to be 20 feet above the ground, subtends an angle of $20^\circ$ at the same point; shew that the height of the house is nearly 22 yards;

$$\log_{10} 200 = 2{\cdot}3010300, \quad L \tan 50^\circ\,15' = 10{\cdot}0800379,$$
$$L \tan 20^\circ = 9{\cdot}5610659, \quad \log_{10} 6{\cdot}606 = 0{\cdot}8200020.$$

## XI. OXFORD AND CAMBRIDGE SCHOOL EXAMINATIONS, 1884.

1. Define a degree, and the unit of circular measure; and find the ratio of the first to the second.

In a triangle $ABC$, the circular measure of one angle ($A$) is $a$, and the number of degrees in another is $B$. If the ratio of the number of degrees in the third angle ($C$) to the circular measure of $B$ be $B : a$, prove that $A$, $B$, $C$ are in continued proportion.

2. Define the secant of an angle, and trace its changes in sign and magnitude as the angle increases from $90^\circ$ to $270^\circ$.

Simplify

$$\left\{\frac{1}{\cos\theta + \tan^2\theta \sin\theta} - \frac{1}{\cos\theta \cot^2\theta + \sin\theta}\right\} \times \left\{\frac{\sec\theta \operatorname{cosec}\theta - 1}{\operatorname{cosec}\theta - \sec\theta}\right\}.$$

3. Prove that if two angles have the same tangent, their difference is a multiple of two right angles.

Solve the equation

$$\sec\theta\,(\sec^2\theta + 2)(\operatorname{cosec}\theta - \sin\theta) = 4.$$

4. Prove geometrically that when $A + B$ is less than $90^\circ$,

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

Prove that

$$\sin^{-1}\frac{1}{3} + \sin^{-1}\frac{1}{3\sqrt{11}} + \sin^{-1}\frac{3}{\sqrt{11}} = \frac{\pi}{2}.$$

5. A hill is inclined at a angle $36^\circ$ to the horizon. An observer walks 100 yards away from the foot of the hill, and then finds that the elevation of a point halfway up the hill is $18^\circ$. Find the height of the hill; and find the ratio of an error in measuring the distance walked to the consequent error in the height of the hill.

6. In any triangle prove that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{1}{2R},$$

where $R$ is the radius of the circle described round $ABC$.

Find $b$, $c$, $R$ in the triangle for which $B = 45^\circ$, $C = 60^\circ$, $a = \sqrt{2}$.

7. Find the area of a regular polygon of $n$ sides; and deduce the area of a circle.

8. Find the cosines of the half angles of a triangle in terms of the sides.

If the bisectors of the angles $A$, $B$, $C$ of a triangle meet the sides in $D$, $E$, $F$ respectively, prove that

$$\frac{(s-b)\,DF^2}{(c+a)^2}-\frac{(s-c)\,DE^2}{(a+b)^2}=(c-b)\left\{\frac{abc}{(b+c)(c+a)(a+b)}\right\}^2.$$

*Questions* 9, 10 *in Higher Trigonometry.*

## XII. OXFORD AND CAMBRIDGE SCHOOL EXAMINATIONS, 1885.

1. What is the unit of circular measure? Why is this unit used?

The angles of a triangle are in A.P.; prove that the mean angle is 60°. If the number of right angles in the greatest is equal to the circular measure of the least, express the angles in degrees

$$\left(\pi=\frac{22}{7}\right).$$

2. Define the tangent of an angle less than 90°, and extend the definition to angles unlimited in magnitude.

Find $\tan 30^0$, $\tan 120^0$, $\tan 240^0$.

3. Find the general solution of the equation $\sin\theta=\sin\alpha$.

Solve the equations:

(i) $2\cos^2\theta+\sqrt{2}\sin\theta=2$.

(ii) $(1+\tan\theta)(1-\sin 2\theta)=1-\tan\theta$.

4. Prove geometrically:

(i) $\sin(A-B)=\sin A\cos B-\cos A\sin B$.

(ii) $\dfrac{\cos(A+B)}{\sin(A-B)}=\dfrac{1-\tan A\tan B}{\tan A-\tan B}$.

Find the relation between $\alpha$, $\beta$ and $\gamma$, in order that

$\cot\alpha\cot\beta\cot\gamma-\cot\alpha-\cot\beta-\cot\gamma$ should vanish.

5. Prove the following statements:

(i) $\dfrac{1+\sin 2x+\cos 2x}{1+\sin 2x-\cos 2x}=\cot x$.

(ii) $3\cos^2\alpha=4\cos^6\dfrac{\alpha}{2}+4\sin^6\dfrac{\alpha}{2}-1$.

(iii) $\cos 3\alpha\cos 2\alpha+\sin 4\alpha\sin\alpha=\cos\alpha\cos 2\alpha$.

6. Prove that in any triangle

$$\cos\left(\frac{A}{2}\right) = \sqrt{\left(\frac{s(s-a)}{bc}\right)}.$$

Having given that

$a = 12270$ ft., $b = 11550$ ft., $c = 11433$ ft.,
$\log 1{\cdot}76265 = {\cdot}2461661$, $\log 5{\cdot}3565 = {\cdot}7288811$,
$\log 1{\cdot}155 = {\cdot}0625820$, $\log 1{\cdot}1433 = {\cdot}0581602$,
$L \cos 32^\circ 15' = 9{\cdot}9272306$ diff. for $1' = {\cdot}0000797$;

find the angle $A$.

7. Find an expression for the radius of one of the described circles of a triangle in terms of the sides.

Prove that

$$\frac{bc}{r_1} + \frac{ca}{r_2} + \frac{ab}{r_3} = 2R\left\{\frac{b}{a} + \frac{c}{a} + \frac{c}{b} + \frac{a}{b} + \frac{a}{c} + \frac{b}{c} - 3\right\}.$$

8. Prove that if $\theta$ be the circular measure of an angle, the limiting value of $\frac{\sin\theta}{\theta}$ as $\theta$ is diminished is unity.

What is the limiting value of $\frac{\sin n'}{n}$ when $n$ is diminished?

*Questions* 9, 10, 11 *in Higher Trigonometry.*

## XIII. TRINITY COLLEGE. *June* 6, 1885. 9—12.

1. Define the sine and tangent of an angle.

The base $BC$ of a triangle $ABC$ is trisected in $Q$ and $R$. Prove that

$$\sin BAR \sin CAQ = 4 \sin BAQ \sin CAR.$$

2. Give a geometrical proof that

(i) $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$,
(ii) $\cos\phi - \cos\theta = 2\sin\frac{1}{2}(\theta+\phi)\sin\frac{1}{2}(\theta-\phi)$,

where $\theta$ and $\phi$ are each less than a right angle.

Show that if

$$\cos(\theta+\phi) = \frac{\sin^2\theta\cos\phi - \cos^2\theta\sin\phi}{\cos\theta\tan\alpha} = \frac{\sin^2\phi\cos\theta - \cos^2\phi\sin\theta}{\cos\phi\tan\beta}.$$

then will

$$\cos(\alpha+\beta) = \frac{\sin^2\alpha\cos\beta - \cos^2\alpha\sin\beta}{\cos\alpha\tan\theta} = \frac{\sin^2\beta\cos\alpha - \cos^2\beta\sin\alpha}{\cos\beta\tan\phi}.$$

3. Find by mathematical induction the value of

$$\tan(\theta_1+\theta_2+\ldots+\theta_n) \text{ in terms of } \tan\theta_1,\ \tan\theta_2,\ldots.$$

Show that in a triangle $ABC$

$$\tan\tfrac{3}{4}A+\tan\tfrac{3}{4}B+\tan\tfrac{3}{4}C=\cot\tfrac{3}{4}A+\cot\tfrac{3}{4}B+\cot\tfrac{3}{4}C,$$

provided $$\tan\tfrac{3}{4}A\tan\tfrac{3}{4}B=\cot\tfrac{3}{4}C.$$

4. From the results

$$a\operatorname{cosec}A=b\operatorname{cosec}B=c\operatorname{cosec}A, \text{ and } A+B+C=\pi,$$

deduce the formulæ

(i) $a^2=b^2+c^2-2bc\cos A$;

(ii) $a=b\cos C+c\cos B$.

In the base $BC$ (produced if necessary) of an isosceles triangle $ABC$ a point $D$ is taken such that the sum of $AD$ and $AB$ is $n$ times $CD$. Show that their difference is $1/n$th of $BD$.

5. The base of a triangle is equal to its altitude, and the two other sides are known. Determine the remaining parts of the triangle by formulæ adapted to logarithms.

Show that if $r$ be the ratio of the two given sides, $r$ must lie between

$$\tfrac{1}{2}(\sqrt{5}-1) \text{ and } \tfrac{1}{2}(\sqrt{5}+1).$$

6. Find the radius of the circle inscribed in a triangle in terms of one side and two angles.

Circles are inscribed in the triangles $D_1E_1F_1$, $D_2E_2F_2$, $D_3E_3F_3$, where $D_1$, $E_1$, $F_1$ are the points of contact of the circles escribed to the side $BC$, &c. Show that if $r_1$, $r_2$, $r_3$ be the radii of these circles

$$\frac{1}{r_1}:\frac{1}{r_2}:\frac{1}{r_3}=1-\tan\tfrac{1}{4}A:1-\tan\tfrac{1}{4}B:1-\tan\tfrac{1}{4}C.$$

7. Prove that the ratio of the radii $r$ and $R$ of the circles inscribed and circumscribed about a triangle $ABC$ is

$$4\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C:1.$$

Two circles of radii $\rho_1$ and $\rho_2$ are drawn through the centre of the circumscribing circle to touch the sides $AB$ and $AC$. Show that

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}-\frac{r}{\rho_1\rho_2}=\frac{r+2R}{R^2}.$$

8. Show that

$$\tan^{-1}x-\tan^{-1}y=\tan^{-1}\frac{1}{y}-\tan^{-1}\frac{1}{x}.$$

Eliminate $a$ between

$$a\cot(\theta-a)=b\cot(\phi-a)=c\cot(\psi-a).$$

*Questions* 9, 10, 11, 12 *in Higher Trigonometry.*

## XIV. JESUS, CHRIST'S, EMMANUEL AND SIDNEY SUSSEX COLLEGES. *June* 9, 1885. 1—4.

1. Define the *sine* and *tangent* of an angle.

A pyramid has for base a square of side $a$; its vertex lies on a line through the middle point of the base, perpendicular to it, and at a distance $h$ from it; prove that the angle $\alpha$ between two lateral faces is given by

$$\sin\alpha=\frac{2h\sqrt{2a^2+4h^2}}{a^2+4h^2}.$$

2. Prove geometrically that

(i) $\cos B-\cos A=2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}.$

(ii) $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}.$

If $\alpha$, $\beta$, $\gamma$ are in Arithmetical Progression, prove that

$$2\left\{\frac{\tan(\alpha+\beta)}{\tan(\beta+\gamma)}-\frac{\tan(\alpha-\beta)}{\tan(\beta-\gamma)}\right\}^{-1}+1=\frac{\sin 4\beta}{\sin(\alpha-\gamma)}.$$

3. Investigate the value of cosine $72^0$.

Solve the equation

$$32\sin^5x+16\cos^4x+12\sin 2x\cos x+20\sin^2x-22\sin x=15.$$

4. Find the value of $\tan 22\frac{1}{2}^\circ$.

Trace the changes in $\dfrac{\tan 3\alpha}{\tan^3\alpha}$ from 0 to $\dfrac{\pi}{2}$, showing that $17+12\sqrt{2}$ is a minimum, and $17-12\sqrt{2}$ is a maximum value.

5. Prove that

$$\sin^3\alpha\sin(\beta-\gamma)+\sin^3\beta\sin(\gamma-\alpha)+\sin^3\gamma\sin(\alpha-\beta)$$
$$+\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha+\beta+\gamma)=0.$$

Also, if $\alpha$, $\beta$, $\gamma$ be the angles of any triangle, prove that

$$\cos 4\alpha(\cos 2\beta-\cos 2\gamma)+\cos 4\beta(\cos 2\gamma-\cos 2\alpha)+\cos 4\gamma(\cos 2\alpha-\cos 2\beta)$$
$$=16\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)\sin\alpha\sin\beta\sin\gamma.$$

6. Prove, with the usual notation, that the area of a triangle is equal to

$$\sqrt{s(s-a)(s-b)(s-c)}.$$

Show that the sum of the areas of the two equilateral triangles, each of which has its vertices at three given distances from a fixed point, is equal to the sum of the equilateral triangles on these distances, and the difference of the aforesaid areas is three times the area of the triangle whose sides are the given distances.

7. If $\Delta$ be the area of any quadrilateral whose semiperimeter is $s$, and sides $a$, $b$, $c$, $d$, prove

$$\Delta^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\frac{\alpha+\gamma}{2},$$

where $\alpha$, $\gamma$ are opposite angles of the figure.

8. Two persons are stationed on two different floors of a house, one vertically above the other, and observe the sun sinking behind a distant tower. From the higher station the sun's lower limb seems to graze the top of the tower. From the lower station the higher limb seems to graze it. Determine the height of the tower in terms of its horizontal distance from the stations, the distance between the stations, and the angular diameter of the sun.

9. Prove that the distance between the centres of the inscribed and circumscribed circles of a triangle is $\sqrt{R^2-2Rr}$, where $r$, $R$ are radii of these circles.

If the sum of the pairs of radii of the escribed circles of a triangle taken in order round the triangle be denoted by $s_1$, $s_2$, $s_3$, and the corresponding differences by $d_1$, $d_2$, $d_3$, prove that

$$d_1d_2d_3 + d_1s_2s_3 + d_2s_3s_1 + d_3s_1s_2 = 0.$$

10. Find an expression for the area of a segment of a circle.

With each corner of a square of side $a$ as centre, and a side as radius, a circle is described, show that the area of the curvilinear quadrilateral formed by the intersection of these four circles has an area

$$a^2\left(1-\sqrt{3}+\frac{\pi}{3}\right).$$

*Questions* 11, 12, 13 *in Higher Trigonometry.*

## XV. MATHEMATICAL TRIPOS. PART I. *May* 21st, 1885. 2—5.

*Questions* 1—6 *Algebra.*

vii. Prove the following formulæ geometrically,

(i) $\sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$,

(ii) $\frac{\pi}{4} = \tan^{-1}\left(\frac{m}{n}\right) - \tan^{-1}\frac{m-n}{m+n}$,

(iii) $\sin^2\alpha + \sin^2\beta = \sin^2(\alpha+\beta) - 2\sin\alpha\sin\beta\cos(\alpha+\beta)$.

viii. If $\cot^{-1}x - \cot^{-1}(x+2) = 15^\circ$, find $x$.

If
$$\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right) = \tan^5\left(\frac{\pi}{4}+\frac{\phi}{2}\right),$$

prove that

$$\sin\psi = 5\sin\phi\,\frac{(1+\alpha^2\sin^2\phi)(1+\beta^2\sin^2\phi)}{\left(1+\frac{1}{\alpha^2}\sin^2\phi\right)\left(1+\frac{1}{\beta^2}\sin^2\phi\right)},$$

and find $\alpha$ and $\beta$.

ix. If $A+B+C=\pi$, prove the formulæ,

(i) $$\frac{\tan A}{\tan B\tan C}+\frac{\tan B}{\tan C\tan A}+\frac{\tan C}{\tan A\tan B}$$
$$=\tan A+\tan B+\tan C-2(\cot A+\cot B+\cot C),$$

(ii) $$\sin\tfrac{1}{2}A+\sin\tfrac{1}{2}B+\sin\tfrac{1}{2}C-1$$
$$=4\sin\tfrac{1}{4}(\pi-A)\sin\tfrac{1}{4}(\pi-B)\sin\tfrac{1}{4}(\pi-C),$$

and if $A+B+C=n\pi$,

(iii) $\sin 2A+\sin 2B+\sin 2C=\pm 4\sin A\sin B\sin C$,

(iv) $\cos 2A+\cos 2B+\cos 2C+1=\mp 4\cos A\cos B\cos C$,

and determine when the upper sign is to be used.

x. Express the area of a triangle in terms of the sides.

Show how to construct the right-angled triangle of minimum area which has its vertices on three parallel lines; and if $a$, $b$ are the distances of the middle line from the other two, show that the hypotenuse makes with the parallel lines an angle

$$\cot^{-1}\frac{a-b}{a+b}.$$

If the given angle of the triangle instead of being a right angle is equal to $\alpha$, find the angle which the side opposite to it makes with the parallel lines when the area is a minimum.

xi. Show how to solve a triangle when two sides and the included angle are given.

If two sides of a triangle are 71 and 25 feet and the contained angle $69^0\ 32'$ calculate the remaining angles and side, and show that if a small error has been made in the measurement of the smaller side it will affect the calculated value of the third side very slightly.

xii. Find the radii of the inscribed and escribed circles of a triangle; and if these are $r$, $r_1$, $r_2$, $r_3$ and that of the circumscribed circle is $R$, prove that $r_1+r_2+r_3-r=4R$.

If $D$, $E$, $F$ are the centres of the escribed circles and $O$ that of the inscribed circle, prove that

$$\frac{EF^2}{r_2r_3}+\frac{FD^2}{r_3r_1}+\frac{DE^2}{r_1r_2}=\frac{OD^2}{rr_1}+\frac{OE^2}{rr_2}+\frac{OF^2}{rr_3}=8\,\frac{R}{r}.$$

# ANSWERS TO THE EXAMPLES.

## I. Page 2.

(1) 80. (2) 10. (3) 16. (4) $109\frac{11}{21}$.

(5) 5 acres. (6) $\frac{1760a}{b}$. (7) $\frac{a\,.\,c}{3}$ yds.

(8) A shilling and a three-penny piece.

## II. Pages 4, 5.

(1) 77440. (2) 6 ft. (3) $\frac{125}{1}$. (4) $6\frac{3}{4}$, $\frac{27}{4}$, $\frac{27}{4}$.

(5) 15·625, $15\frac{5}{8}$, $\frac{125}{8}$, $\frac{125}{8}$. (6) $9\frac{9}{10}$, 9 shillings.

(7) $\frac{80a}{b}$, $\frac{80a}{c}$, $\frac{80a}{d}$, $\frac{80a}{k}$. (8) 21 shillings.

## III. Pages 7, 8.

(1) 10 ft. (2) 80 yds. (3) 20 ft. (4) 50 ft.

(5) 90 ft. (6) $20\frac{10}{27}$ nearly. (7) $5a$ feet.

(8) $12a$ yards. (10) $\frac{\sqrt{2}}{2}\,a$ yards. (12) $\frac{2\sqrt{3}}{3}\,a$ feet.

(13) $1 : \sqrt{2}$. (14) $\sqrt{84}$ ft. (15) $2\sqrt{9a^2-b^2}$ ft.

## IV. Page 13.

(1) 9·899 ft. (2) 2489 yds. (3) 58·36 in.

(4) 1138 in. (5) 933·4 ft. (6) 2504 in.

(7) 8·66 ft. (8) 4·607 ft. (9) 48 in.

(10) 85 ft. 11 in.

## V. Pages 21, 22.

(1) $3\frac{1}{7}$ yds. (2) $25\frac{1}{7}$ ft. (3) $150\frac{6}{7}$ in. (4) $3\frac{2}{11}$ ft.

(5) $7\frac{7}{11}$ ft. (6) 560. (7) $15\frac{1}{2}$ nearly. (8) 33600.

(9) $32\frac{1}{7}$. (10) 7 ft. (11) $553\frac{1}{7}$, 13·8 in. (12) $339\frac{3}{7}$ ft.

(13) 443 in. (14) 235 in. (15) 203 in.

(16) 274 in. (17) 1886 in.

## VII. Page 28.

(1) ·632118 of a right angle. (3) ·021827 of a right angle.

(2) 1·0426991 ,, (4) ·03294894 ,,

(5) ·006241 of a right angle. (9) ·6900071 of a right angle.
(6) 10·000812 ,, (10) 1·19030045 ,,
(7) ·3204052 ,, (11) 10·061801 ,,
(8) ·0102034 ,, (12) ·0226048 ,,

(13) $36^{g}\ 78^{\backprime}\ 91^{\backprime\backprime}$. (14) $104^{g}\ 30^{\backprime}\ 21^{\backprime\backprime}$. (15) $1^{g}\ 20^{\backprime}\ 3^{\backprime\backprime}$.
(16) $10^{\backprime}\ 20^{\backprime\backprime}$. (17) $6^{g}\ 25^{\backprime}$. (18) $302^{g}\ 12^{\backprime}\ 50^{\backprime\backprime}$.
(19) $100^{g}\ 10^{\backprime}$. (20) $1^{g}\ 1^{\backprime}\ \cdot 1^{\backprime\backprime}$. (21) $645^{g}\ 10^{\backprime}$.
(22) $2^{g}\ 30^{\backprime}$. (23) $1^{\backprime}\ 10^{\backprime\backprime}$. (24) $10^{\backprime\backprime}$.

## VIII. Page 30.

(1) ·09175 of a right angle $= 9^{g}\ 17^{\backprime}\ 50^{\backprime\backprime}$.
(2) ·0675 ,, $= 6^{g}\ 75^{\backprime}$.
(3) 1·07875 ,, $= 107^{g}\ 87^{\backprime}\ 50^{\backprime\backprime}$.
(4) $\cdot 180429\dot{0}1234567\dot{9}$ $= 18^{g}\ 4^{\backprime}\ 29^{\backprime\backprime}$, etc.
(5) $1\cdot 46\dot{7}$ ,, $= 146^{g}\ 77^{\backprime}\ 77\cdot\dot{7}^{\backprime\backprime}$.
(6) $\cdot 5\dot{4}$ ,, $= 54^{g}\ 44^{\backprime}\ 44\cdot\dot{4}^{\backprime\backprime}$.
(7) $1^{0}\ 14'\ 15''$. (8) $7^{0}\ 52'\ 30''$. (9) $153^{0}\ 24'\ 29\cdot 34''$.
(10) $21^{0}\ 36'\ 8\cdot 1''$. (11) $16^{0}\ 12'\ 37\cdot 26''$. (12) $31^{0}\ 30'$.

## IX. Page 35.

**1.** (1) 2 right angles or $180^{0}$. (2) $\frac{3}{5}$ of a right angle.
(3) $\frac{2}{\pi}$ right angles. (4) $\frac{6}{\pi}$ right angles.
(5) 2 right angles. (6) $\frac{4}{\pi^{2}}$ right angles.
(7) $\frac{2\theta}{\pi}$ right angles. (8) ·002 of a right angle.
(9) 20 right angles.

**2.** (1) $\pi$. (2) $2\pi$. (3) $\frac{\pi}{3}$. (4) $\frac{\pi}{8}$.
(5) $\frac{\pi}{180}$. (6) $1^{c}$. (7) $\frac{n}{180}\pi$. (8) $\frac{1}{2}^{c}$. (9) $\frac{A\pi}{180}$.

**3.** (1) $\frac{\pi}{6}$. (2) $\frac{\pi}{4}$. (3) $\frac{\pi}{12}$. (4) $\frac{\pi}{200}$.
(5) $\frac{\pi}{20000}$. (6) $\frac{\pi}{200000}$. (7) $\frac{n\pi}{200}$. (8) $1^{c}$. (9) $5\pi$.

**4.** (1) $\frac{1}{3}$. (2) $\frac{10}{9}$. (3) 1. (4) $\frac{3\pi}{50}$. (5) $\frac{33}{25}$. (6) $\frac{\pi}{180}$

## X. Pages 37—39.

(1) $\frac{3}{5}$. (2) 90. (3) $4\frac{4}{5}$. (4) $\frac{700}{11}$.

(5) $112\frac{1}{2}$ ft. (6) $5\frac{37}{88}$ ft. (7) $9\frac{3}{7}$ ft.

(8) 838000 miles. (9) $\frac{1}{9}$ radian $= 6\frac{4}{11}$ degrees.

(10) $21\frac{21}{22}$ degrees. (11) $51\frac{9}{11}''$. (12) about 34 yds.

(13) 1 : 3·1416. (14) 3·1416. (15) $3{\cdot}141\dot{6}$.

(16) 400 : 1. (17) ·0000484.... (18) $49\frac{1}{11}$ in.

(19) $\frac{\pi}{2}$ i.e. a right angle. (21) 473 : 489.

(22) (i) $k=1$, (ii) $k=\frac{180}{\pi}$.

## XI. Pages 40—43.

(1) $\frac{\pi}{10}$. (2) $33^0$, $15^0$. (3) $90^0$, $45^0$. (4) $18^0$, $22\frac{1}{2}^0$.

(5) $38^0$, $18^0$. (7) $45^0$, $60^0$, $75^0$. (8) $39^0$, $60^0$, $81^0$.

(9) $33{\cdot}\dot{3}^g$, $66{\cdot}\dot{6}^g$, $100^g$. (14) $25^0$, $8^0\ 6'$. (15) $\frac{21}{25}$.

(16) $\frac{n\pi}{10800}$. (17) (i) $120^0$, $133{\cdot}\dot{3}^g$, $\frac{2\pi}{3}$, (ii) $135^0$, $150^g$, $\frac{3\pi}{4}$,

(iii) $156^0$, $173{\cdot}\dot{3}^g$, $\frac{13\pi}{15}$. (20) $450'$. (21) (i) $3\frac{3}{4}$, (ii) $\frac{15}{2\pi}$.

(22) $\frac{18}{15}^0$. (23) a right angle. (24) $\frac{10ac}{9b}$.

(25) $\frac{9a+10b}{10c}$ degrees. (26) $\frac{5\pi}{9}$. (27) $\frac{5\pi}{19}$, $\frac{9\pi}{38}$, $\frac{\pi}{2}$.

(28) $\frac{1800\pi}{19\pi+1800}$. (29) $\frac{\pi^2}{10\pi+18000}$. (30) 9 or 16.

## XII. Pages 50—52.

(1) (i) $DA, BD$. (ii) $DB, AD$. (iii) $DA, CD$. (iv) $DC, AD$.

(2) (i) $\frac{DB}{AB}$. (ii) $\frac{DC}{CA}$. (iii) $\frac{CD}{AD}$. (iv) $\frac{DA}{BA}$. (v) $\frac{DB}{AD}$.

(vi) $\frac{DC}{AC}$. (vii) $\frac{CD}{CA}$. (viii) $\frac{DA}{CD}$. (ix) $\frac{BD}{BA}$. (x) $\frac{DA}{CA}$.

(3) (i) $\frac{DB}{CB}$, $\frac{BA}{CA}$. (ii) $\frac{CD}{CB}$, $\frac{CB}{CA}$. (iii) $\frac{DB}{CD}$, $\frac{BA}{CB}$.

(iv) $\frac{DB}{AB}$, $\frac{BC}{AC}$. (v) $\frac{AD}{AB}$, $\frac{AB}{AC}$. (vi) $\frac{DB}{AD}$, $\frac{BC}{AB}$.

(4) (i) $\frac{DA}{BA}$. (ii) $\frac{BA}{EA}$ or $\frac{AC}{EC}$. (iii) $\frac{DC}{BC}$. (iv) $\frac{AB}{AE}$.

(v) $\frac{AD}{AB}$ or $\frac{AB}{AC}$. (vi) $\frac{BD}{BC}$. (vii) $\frac{DB}{CD}$, or $\frac{BA}{CB}$, or $\frac{AE}{CA}$.

(viii) $\frac{DA}{BD}$. (ix) $\frac{BA}{EB}$ or $\frac{AC}{EA}$. (x) $\frac{DC}{BD}$. (xi) $\frac{DB}{AB}$ or $\frac{BC}{AC}$.

(xii) $\frac{BE}{AE}$.

(5) $\sin A = \frac{3}{5}$, $\cos A = \frac{4}{5}$, $\tan A = \frac{3}{4}$; $\sin B = \frac{4}{5}$, $\cos B = \frac{3}{5}$, $\tan B = \frac{4}{3}$.

(7) Of the smaller angle, the sine $= \frac{5}{13}$, cosine $= \frac{12}{13}$, tangent $= \frac{5}{12}$.
Of the larger angle, the sine $= \frac{12}{13}$, cosine $= \frac{5}{13}$, tangent $= \frac{12}{5}$.

(8) Of the smaller angle, the sine $= \frac{1}{2}$, cosine $= \frac{\sqrt{3}}{2}$, tangent $= \frac{1}{\sqrt{3}}$.

Of the larger angle, the sine $= \frac{\sqrt{3}}{2}$, cosine $= \frac{1}{2}$, tangent $= \sqrt{3}$.

## XIV. Pages 63—65.

(1) 179 ft. (2) 346 ft. (3) 86·6 ft. (4) 138·5 ft.
(5) $7\frac{1}{2}$ ft. (6) 60°, 173 ft. (7) 63·17 yds. (8) 34·15 ft.
(9) 73·2 ft. (10) 86·6 ft. (11) ·866 miles = 1524 yds.
(12) 173·2 yds. (14) 373 ft. (15) 3733 ft.
(16) $\frac{\sqrt{6}}{2}$ miles = 6465 ft. (17) $\frac{\sqrt{3}\,.\,a}{3b}$. (18) 30°.
(19) about 523·6 miles.

## XVI. Page 74.

(1) $\sin A = \sqrt{1-\cos^2 A}$, $\tan A = \frac{\sqrt{1-\cos^2 A}}{\cos A}$, $\cot A = \frac{\cos A}{\sqrt{1-\cos^2 A}}$,

$\sec A = \frac{1}{\cos A}$, $\operatorname{cosec} A = \frac{1}{\sqrt{1-\cos^2 A}}$.

(2) $\sin A = \frac{1}{\sqrt{1+\cot^2 A}}$, $\cos A = \frac{\cot A}{\sqrt{1+\cot^2 A}}$, $\tan A = \frac{1}{\cot A}$,

$\sec A = \frac{\sqrt{1+\cot^2 A}}{\cot A}$, $\operatorname{cosec} A = \sqrt{1+\cot^2 A}$.

(3) $\sin A = \frac{\sqrt{\sec^2 A - 1}}{\sec A}$, $\cos A = \frac{1}{\sec A}$, $\tan A = \sqrt{\sec^2 A - 1}$,

$\cot A = \frac{1}{\sqrt{\sec^2 A - 1}}$, $\operatorname{cosec} A = \frac{\sec A}{\sqrt{\sec^2 A - 1}}$.

(4) $\sin A = \frac{1}{\operatorname{cosec} A}$, $\cos A = \frac{\sqrt{\operatorname{cosec}^2 A - 1}}{\operatorname{cosec} A}$, $\tan A = \frac{1}{\sqrt{\operatorname{cosec}^2 A - 1}}$,

$\cot A = \sqrt{\operatorname{cosec}^2 A - 1}$, $\sec A = \frac{\operatorname{cosec} A}{\sqrt{\operatorname{cosec}^2 A - 1}}$.

(5) $\cos A = \sqrt{1 - \sin^2 A}$, $\tan A = \frac{\sin A}{\sqrt{1 - \sin^2 A}}$, $\cot A = \frac{\sqrt{1 - \sin^2 A}}{\sin A}$

$\sec A = \frac{1}{\sqrt{1 - \sin^2 A}}$, $\operatorname{cosec} A = \frac{1}{\sin A}$.

(6) $\sin A = \frac{\tan A}{\sqrt{1 + \tan^2 A}}$, $\cos A = \frac{1}{\sqrt{1 + \tan^2 A}}$, $\cot A = \frac{1}{\tan A}$,

$\sec A = \sqrt{1 + \tan^2 A}$, $\operatorname{cosec} A = \frac{\sqrt{1 + \tan^2 A}}{\tan A}$.

## XVII. Page 75.

(1) $\frac{3}{4}$, $\frac{5}{3}$. (2) $\frac{2\sqrt{2}}{3}$, $\frac{1}{2\sqrt{2}}$. (3) $\frac{4}{5}$, $\frac{5}{3}$.

(4) $\frac{1}{\sqrt{15}}$, $\frac{\sqrt{15}}{4}$. (5) $\frac{\sqrt{3}}{2}$, $\frac{1}{2}$. (6) $\frac{\sqrt{5}}{3}$, $\frac{2}{3}$.

(7) $\frac{b}{\sqrt{c^2 - b^2}}$. (8) $\frac{a}{\sqrt{a^2 + b^2}}$, $\frac{b}{\sqrt{a^2 + b^2}}$.

(9) $\frac{\sqrt{a^2 - 1}}{a}$, $\frac{1}{\sqrt{a^2 - 1}}$. (11) $h^2(1 + k^2) = 1$.

## XVIII. Page 77.

(2) $\sec \theta$ increases continuously from 1 to $\infty$.
(3) $\sin A$ diminishes continuously from 1 to 0.
(4) $\cot \theta$ diminishes continuously from $\infty$ to 0.

## XIX. Page 82.

(4) Yes. (5) No. (6) Yes.
(7) (i) $60^\circ$. (ii) $-100^\circ$. (iii) $0^\circ$. (iv) $-260^\circ$. (v) $115^\circ$.
(vi) $410^\circ$. (vii) $-\frac{\pi}{4}$. (viii) $\frac{2\pi}{3}$.

## XX. Page 84.

(1) $45^\circ$. (2) $30^\circ$. (3) $45^\circ$. (4) $60^\circ$.
(5) $30^\circ$. (6) $30^\circ$. (7) $30^\circ$. (8) $0^\circ$, or $45^\circ$.

(9) $90^\circ$, or $60^\circ$. (10) $60^\circ$. (11) $45^\circ$. (12) $45^\circ$.
(13) $90^\circ$, or $45^\circ$. (14) $18^\circ$. (15) $45^\circ$. (16) $45^\circ$.
(17) $30^\circ$. (18) $30^\circ$.

### XXI. Page 85.

(3) the value 3 is inadmissible. (4) $\frac{2 \pm \sqrt{2}}{4}$.

(5) $\frac{3}{4}$, or $\frac{1}{2}$. (6) $\frac{3}{4}$, or $\frac{1}{3}$.

(7) the value $-\frac{7\sqrt{3}}{6}$ is inadmissible. (9) $1 - \sin^4 \theta$.

(10) $1 - 3\sin^2\theta + 3\sin^4\theta$. (11) $\frac{1 - 2\cos^2\theta + 2\cos^4\theta}{\cos^4\theta}$.

(13) $\frac{1 - \sin A}{1 + \sin A}$.

(14) $\operatorname{cosec}\theta$ decreases continuously from $\infty$ to 1.

(15) $\cot\theta$ increases continuously from 0 to $\infty$.

(16) $\theta = \frac{\pi}{4}$, $\phi = \frac{\pi}{12}$.

### XXII. Page 89.

(1) $+6$. (2) 0. (3) $+2$. (4) $+3$.
(5) $+10$. (6) 0. (7) $+7$. (8) $+7$.

### XXIV. Page 94.

(1) The second. (2) The fourth. (3) The second.
(4) The third. (5) The fourth. (6) The first.
(7) The second. (8) The first. (9) The first.
(10) The fourth. (11) The fourth.
(12) The first, if $n$ be even, the third, if $n$ be odd.

### XXV. Page 98.

(1) $+, +, +$. (2) $+, -, -$. (3) $-, -, +$.
(4) $-, +, -$. (5) $-, +, -$. (6) $-, -, +$.
(7) $+, -, -$. (8) $+, +, +$. (9) $+, +, +$.
(10) $+, +, +$. (11) $+, -, -$. (12) $-, +, -$.

### XXVI. Page 100.

(1) $+\frac{1}{2}, -\frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}$. (2) $+\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -1$.

(3) $+\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\sqrt{3}$. (4) $-\frac{1}{2}, +\frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}$.

(5) $-\frac{1}{\sqrt{2}}, +\frac{1}{\sqrt{2}}, -1.$

(6) $+\frac{\sqrt{3}}{2}, +\frac{1}{2}, +\sqrt{3}.$

(7) $-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, +1.$

(8) $-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, +1.$

(9) $+\frac{1}{2}, +\frac{\sqrt{3}}{2}, +\frac{1}{\sqrt{3}}.$

(10) $+\frac{1}{2}, +\frac{\sqrt{3}}{2}, +\frac{1}{\sqrt{3}}.$

(11) $-\frac{\sqrt{3}}{2}, -\frac{1}{2}, +\sqrt{3}.$

(12) $-\frac{\sqrt{3}}{2}, +\frac{1}{2}, -\sqrt{3}.$

(13) $+\frac{1}{\sqrt{2}}, +\frac{1}{\sqrt{2}}, +1.$

(14) $\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\sqrt{3}.$

(15) $-\frac{1}{2}, -\frac{\sqrt{3}}{2}, +\frac{1}{\sqrt{3}}.$

## XXVII. Page 100.

Each of these expressions changes *continuously* as the angle $A$ increases from $0^0$ to $360^0$, and their values are *repeated* at each complete revolution of the revolving line.

Their values are given below when they are zero and when they cease to increase and begin to decrease, and vice versa. The first table gives also the sign of each between the values.

| | $0^0$ | | $90^0$ | | $180^0$ | | $270^0$ | |
|---|---|---|---|---|---|---|---|---|
| (1) $\cos A$ | $+1$ | + | 0 | − | $-1$ | − | 0 | + |
| (2) $\tan A$ | 0 | + | $\infty$ | − | 0 | + | $\infty$ | − |
| (3) $\cot A$ | $\infty$ | + | 0 | − | $\infty$ | + | 0 | − |
| (4) $\sec A$ | $+1$ | + | $\infty$ | − | $-1$ | − | $\infty$ | + |
| (5) $\operatorname{cosec} A$ | $\infty$ | + | $+1$ | + | $\infty$ | − | $-1$ | − |
| (6) $1-\sin A$ | $+1$ | + | 0 | + | $+1$ | + | 2 | + |
| (7) $\sin^2 A$ | 0 | + | $+1$ | + | 0 | + | $+1$ | + |

| | $0^0$ | $45^0$ | $90^0$ | $135^0$ | $180^0$ | $225^0$ | $270^0$ | $315^0$ |
|---|---|---|---|---|---|---|---|---|
| (8) $\sin A \,.\, \cos A$ | 0 | $+\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | 0 | $+\frac{1}{2}$ | 0 | $-\frac{1}{2}$ |
| (9) $\sin A + \cos A$ | $+1$ | $+\sqrt{2}$ | $+1$ | 0 | $-1$ | $-\sqrt{2}$ | $-1$ | 0 |
| (10) $\tan A + \cot A$ | $\infty$ | $+2$ | $\infty$ | $-2$ | $\infty$ | $+2$ | $\infty$ | $-2$ |
| (11) $\sin A - \cos A$ | $-1$ | 0 | $+1$ | $+\sqrt{2}$ | $+1$ | 0 | $-1$ | $-\sqrt{2}$ |

The following figures exhibit the changes in the sign and magnitude of $\sin\theta$ (fig. i.), $\cos\theta$ (fig. ii.), and $\tan\theta$ (fig. iii.). The measure of the distance from 0 along the line $0X$ = the circular measure of the angle; the vertical distance from $0X$ measures the Ratio.

Fig. i. The Curve of the Sine.

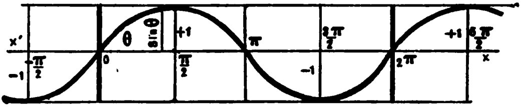


Fig. ii. The Curve of the Cosine.

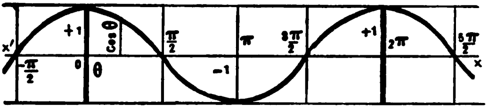
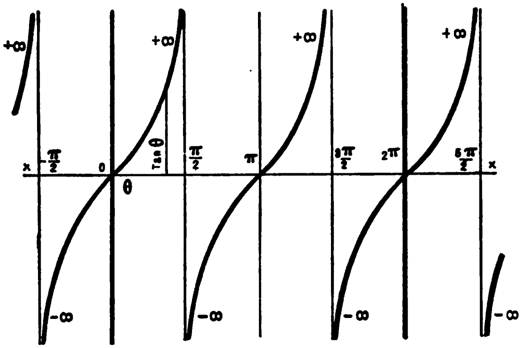


Fig. iii. The Curve of the Tangent.



## XXXI. Page 112.

(1) (i) $30^\circ, 150^\circ, -210^\circ, -330^\circ$. (ii) $45^\circ, 135^\circ, -225^\circ, -315^\circ$. (iii) $60^\circ, 120^\circ, -240^\circ, -300^\circ$. (iv) $-30^\circ, -150^\circ, 210^\circ, 330^\circ$

(2) (i) $20^\circ, 160^\circ, 380^\circ, 520^\circ$. (ii) $\frac{5\pi}{4}, \frac{7\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}$.

(iii) $\frac{8\pi}{7}, \frac{13\pi}{7}, \frac{22\pi}{7}, \frac{27\pi}{7}$.

(3) (i) $\theta = n\pi + (-1)^n\left(-\frac{\pi}{6}\right)$. (ii) $\theta = n\pi + (-1)^n\frac{\pi}{6}$.

(iii) $n\pi \pm \frac{\pi}{4}$.

## XXXII. Page 115.

(1) (i) $\theta = 2n\pi \pm \frac{1}{3}\pi$. (ii) $\theta = n\pi + \frac{1}{4}\pi$. (iii) $\theta = n\pi - \frac{1}{4}\pi$.
(iv) $\theta = n\pi - \frac{1}{3}\pi$. (v) $\theta = 2n\pi \pm \frac{4}{5}\pi$. (vi) $\theta = n\pi + \frac{3}{4}\pi$.
(3) The tangent. (4) No.
(5) (i) $60^0$, $-60^0$, $300^0$, $-300^0$. (ii) $45^0$, $225^0$, $405^0$, $-135^0$.
(iii) $-45^0$, $-225^0$, $135^0$, $315^0$. (iv) $-60^0$, $-240^0$, $120^0$, $300^0$.
(v) $144^0$, $-144^0$, $216^0$, $-216^0$. (vi) $135^0$, $-45^0$, $-225^0$, $315^0$

## XXXIV. Page 120.

(5) 1, $\frac{24}{25}$. (6) $\frac{16}{65}$, $\frac{33}{65}$.

## XXXVIII. Page 131.

(1) $\sin(\theta + \phi) + \sin(\theta - \phi)$. (2) $\cos(\alpha - \beta) + \cos(\alpha + \beta)$.
(3) $\sin(2\alpha + 3\beta) + \sin(2\alpha - 3\beta)$. (4) $\cos 2\alpha + \cos 2\beta$.
(5) $\sin 8\theta - \sin 2\theta$. (6) $\cos\theta + \cos 2\theta$. (7) $\frac{1}{2}(\cos 3\theta - \cos 5\theta)$.
(8) $\frac{1}{2}(\sin 4\theta - \sin\theta)$. (9) $\sin 60^0 + \sin 40^0$.
(10) $\frac{1}{2}(\sin 60^0 - \sin 30^0)$. (11) $2\cos 3\theta \cos 2\theta$.
(12) $-\cos 4\theta \sin 2\theta$. (13) $4\cos^2\frac{\theta}{2}\sin 2\theta$.

## XXXIX. Page 132.

(4) $\tan 2\alpha$. (5) $\tan x$. (7) $-2\cot 2x$. (8) $\tan^2 A$.
(12) $\sin 2\theta = 1$, $\therefore \theta = 45^0$; $\sin 2\phi = \frac{1}{2}$, $\therefore \phi = 15^0$.

## XLV. Page 147.

(2) $\sin 2A = \frac{4}{5}$, $\sin 3A = -\frac{11}{25}\sqrt{5}$.
(3) $\sin 2\theta = -\frac{1}{8}\sqrt{15}$, $\sin 3\theta = \frac{3}{16}\sqrt{15}$, $\cos 3\theta = -\frac{11}{16}$; in the second.

## XLVI. Page 149.

(3) $\sin\frac{A}{2} = +\sqrt{\frac{1-\cos A}{2}}$. (5) $\sin\frac{A}{2} = +\sqrt{\frac{1-\cos A}{2}}$.

## XLVII. Page 154.

(1) (i) +, −. (ii) −, +. (iii) −, −. (iv) +, −. (v) +, −.
(vi) +, +. (vii) −, −. (viii) −, −.
(3) $\sin 9^0 = \frac{1}{4}\{\sqrt{(3+\sqrt{5})} - \sqrt{(5-\sqrt{5})}\}$,
$\cos 9^0 = \frac{1}{4}\{\sqrt{(3+\sqrt{5})} + \sqrt{(5-\sqrt{5})}\}$,
$\sin 81^0 = \cos 9^0$, $\cos 189^0 = -\cos 9^0$, $\tan 202\frac{1}{2}^0 = \sqrt{2} - 1$,
$\tan 97\frac{1}{2}^0 = -(\sqrt{3} + \sqrt{2})(\sqrt{2} + 1)$.
(7) Between $(2n \times 180^0 - 45^0)$ and $(2n \times 180^0 + 45^0)$.

## XLVIII. Page 157.

(1) $\theta = -\frac{\pi}{4} + n\pi + (-1)^n \frac{\pi}{6}$. (2) $\theta = -\frac{\pi}{3} + n\pi + (-1)^n \frac{\pi}{6}$.

(3) $\theta = -\frac{\pi}{4} + n\pi + (-1)^n \frac{\pi}{3}$. (4) $\theta = +\frac{\pi}{4} + n\pi + (-1)^n \frac{\pi}{4}$.

(5) $\theta = +\frac{\pi}{4} + 2n\pi \pm \frac{\pi}{4}$. (6) $\theta = \frac{\pi}{6} + n\pi \pm \frac{\pi}{4}$.

(7) $x = -68^\circ 12' + n \times 180^\circ + (-1)^n (21^\circ 48')$.

(8) $x = -69^\circ 26' 30'' + 2n \times 180^\circ \pm (69^\circ 26' 30'')$.

(9) $x = 75^\circ 4' + n \times 180^\circ + (-1)^n (14^\circ 56')$.

(10) $x = -\alpha - \frac{\pi}{4} + 2n\pi \pm \beta$.

## L. Pages 159, 160.

(7) $\pm(\sqrt{3}+1)$. (8) $\pm\sqrt{2}$. (9) $\pm 1$.

(13) $\frac{1}{3}(2n+1)\pi$; or, $\frac{1}{3}(3n \pm 1)\pi$. (14) $n\pi \pm \frac{\pi}{3}$; or $\frac{1}{6}(2n+1)\pi$.

(15) $\frac{1}{6}(2n+1)\pi$; or $\frac{n\pi}{3} + (-1)^n \frac{\pi}{12}$.

(16) $(2n+1)\pi$; or, $\cos^{-1}\frac{a-b}{a}$. (17) $\frac{n\pi}{m}$; or $\frac{2r+1}{m \pm n} \cdot \pi$.

(18) Both equations are satisfied if $(x^2 - y^2) =$ an odd integer; or, if $(x+y)^2 = 2n$, and $(x-y)^2 = 4m - 2n$, $m$ and $n$ being integers.

(19) (i) $2 \sin\theta . \cos\theta = \sin 2\theta$, and therefore it goes through the same changes as $\sin\theta$, while $\theta$ changes from 0 to $2\pi$. (ii) $\cos^2\theta - \sin^2\theta = \cos 2\theta$. (iii) $\sin 3\theta$ goes through the same changes as $\sin\theta$, while $\theta$ goes from 0 to $3\pi$. (iv) $\cot 2A$; compare with $\cot A$. (v) $\sin(\theta + \alpha)$ goes through the same changes while $\theta$ goes from 0 to $\pi$, as $\sin\theta$ goes through while $\theta$ goes from $\alpha$ to $\pi + \alpha$. (vi) $\cos(2\theta - \alpha)$ goes through the same changes while $\theta$ goes from 0 to $\pi$, as $\cos 2\theta$ goes through as $\theta$ changes from $-\frac{\alpha}{2}$ to $\pi - \frac{\alpha}{2}$. (See Ans. to XXVII.)

(20) They are the solutions of the equations $\sin\theta = \sin\alpha$, and $\cos\left(\frac{\pi}{2} - \theta\right) = \cos\left(\frac{\pi}{2} - \alpha\right)$; and we know that $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$.

(21) They are the solutions of the equations $\sin\left(\theta + \frac{\pi}{4}\right) = \sin\frac{\pi}{6}$, and $\cos\left(\theta - \frac{\pi}{4}\right) = \cos\frac{\pi}{3}$; also $\cos\left(\theta - \frac{\pi}{4}\right) = \sin\left(\theta + \frac{\pi}{4}\right)$, $\sin\frac{\pi}{6} = \cos\frac{\pi}{3}$.

## LI. Page 162.

(1) (i) $a^{2h+3k}$, (ii) $a^{4h-5k}$, (iii) $a^{\frac{4h}{3}+\frac{5k}{3}}$. (iv) $a^{\frac{5h}{2}+\frac{3k}{2}}$.
(2) (i) 5·4690116. (ii) 10·6243928. (iii) 13·7509366.
(iv) ·8853661. (v) 1·7968680. (vi) 8·9699598.
(vii) 2·7345058.
(3) $2^3, 2^5, 2^{-1}, 2^{-4}, 2^{-3}, 2^7$. (4) $3^2, 3^4, 3^{-1}, 3^{-3}, 3^{-2}, 3^{-4}$.

## LII. Page 163.

(1) ·60206, ·9542426, ·90309, ·7781513, 1·20412, 1·690196.
(2) 1·146128, 1·20412, 1·2552726, 1·3802113, 1·4313639, 1·6232493.
(3) 1, ·69897, 1·1760913, 1·39794, 1·4771213, 1·5440680.
(4) 1·5563026, 1·60206, 1·6812413, 1·69897, 2·30103, 3.
(5) 7·201593, 3·858708. (6) ·7545579, 2·989843.
(7) 1·4532. (8) 2408·6. (9) (i) 4·5868. (ii) ·93646.
(10) 3·9549. (11) 40975·3 sq. ft. (12) 34·925 in.
(13) 3·2617 in. (14) 110115 cub. yds.

## LIII. Page 165.

(1) 3, $\frac{10}{3}$, $\frac{1}{4}$, $\frac{2}{3}$, $-\frac{5}{2}$. (2) 3, 6, $-1$, $-3$, $-6$, 2.
(3) 2, 4, $-1$, $-3$, $-2$, $-4$. (4) $\frac{3}{2}$, $\frac{2}{3}$, $-\frac{1}{4}$, $-1$.
(5) 3, $-1$, 5, $-2$, 3, $-3$. (6) $\frac{4}{3}$, $\frac{2}{3}$, $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{2}$.
(7) ·7781513, 1·6232493, 1·20412.
(8) 1·6901960, 1·5563026, 1·7993406.
(9) 2·30103, 2·7781513, 1·845098.
(10) ·69897, 5228787, 1·69897.
(11) 1·544068, 2·1760913, $-1+$·30103.
(12) ·5440680, ·8627278, $-2+$·9084852.

## LIV. Pages 168, 169.

(1) 4, 2, 0, 5, 1. (2) $-2$, $-5$, $-1$, $-3$.
(3) 3, $-1$, 0, 1, 0, $-7$. (4) 4, 1, 6, 3.
(5) the second decimal place, the first dec. pl., the sixth dec. pl.
(6) ten thousands, units, hundreds, third dec. pl., first dec. pl., units.
(7) 10, 4, 25, 31. (8) 9, 11, 85, 4, 9, 6.
(9) units, fourth dec. pl., thousands, seventh dec. pl., second dec. pl.
(10) tenth integral pl., twelfth dec. pl., fifth dec. pl., units, twelfth dec. pl., first dec. pl.

## LV. Pages 171, 172.

(1) 2·8901023, ·8901023, $\bar{4}$·8901023, 5·8901023.

(2) 6·7714552, ·7714552, $\bar{4}$·7714552, 2·7714552, 3·7714552.

(3) ·27724... (4) ·00001638... (5) ·77448... (6) ·005963...

## LVI. Page 174.

(1) Divide each log by 3. (2) Multiply each log by 2.

(3) Divide each log by ·30103.

(4) Multiply each log by ·4771213.

(5) Divide each log by ·4771213. (6) 3·32190...

(7) 1·183... (8) 1·10730... ·66438...

## LVII. Pages 175, 176.

(1) 3, 0, $\frac{1}{3}$, 0, $\frac{7}{5}$. (4) $\bar{1}$·8121177, 55.

(5) ·51375. (6) 7, 4, 3, 3.

(7) (i) $x=\dfrac{2\log 7}{\log 2+4\log 3}$. (ii) $x=\dfrac{7\log 2+4\log 7}{2\log 3+\log 7}$.

(iii) $x=\dfrac{2\log 7}{2\log 2+\log 3}$. (iv) $x=\dfrac{4(\log 3+\log 7)}{8\log 2+3(\log 3+\log 7)}$.

(8) $2+\dfrac{1}{\log_{10}7}$. (9) $\frac{3}{2}+\dfrac{1}{2\log_{10}3}$. (10) $\dfrac{1}{1-\log_{10}2}$.

(11) $0,\ \dfrac{1}{b+1},\ \dfrac{3a}{2b+2},\ \dfrac{2}{b+1},\ \dfrac{b}{b+1},\ \dfrac{3a+2}{2b+2},\ \dfrac{bc}{b+1}$.

(12) $63-31=32$. (13) $(a^{11}-a^{10})$ integers. (14) 1·9485 nearly

(19) 2·53855. (20) 4·59999. (21) 167 years.

## LVIII. Pages 181, 182.

(1) ·8839066. (2) 2·7513738. (3) $\bar{4}$·9413333.

(4) 6·8086920. (5) ·5710750. (6) 3·70404.

(7) 45740·26. (8) 2492837. (9) ·000439658. (10) 5·689158.

## LIX. Pages 184, 185.

(1) ·6737652. (2) ·6737652 (3) ·9306572.

(4) $41^0\,48'\,37''$. (5) $70^0\,31'\,43{\cdot}6''$. (6) $75^0\,31'\,21''$

(7) 9·8515594. (8) 9·7114477. (9) 10·1338768.

(10) $35^0\,4'\,23''$. (11) $28^0\,16'\,27{\cdot}5''$. (12) $21^0\,56'\,41''$

## LX. Pages 188, 189.

(1) 34° 19′ 31·8″. (2) 1498·2 ft. (3) 45° 36′ 56″.
(4) 5293·4 ft., 6982·3 ft. (5) 576·2 chains.
(6) 4729 chains. (7) 3666·8 feet. (8) 42° 15′, 11444 chains.

## LXI. Page 190.

(1) 3842·9 ft. (2) 281·74 ft. (3) 115·3 ft. (4) 285·6 ft.
(5) 58° 17′ 24″, 31° 42′ 36″. (6) 656·1 chains, 41° 17′ 12″.
(7) 81 ft. (8) 1942 ft. (9) 646·7 miles. (10) 1000 ft.

## LXVI. Pages 208, 209.

(1) 41° 16′ 51·5″. (2) 73° 32′ 12″, 62° 46′ 18″.
(3) 29° 17′ 16″, 31° 55′ 31″. (4) 64° 31′ 58″.
(5) 73°, 23′ 54·4″. (6) 41° 24′ 34·6″. (7) 82° 49′ 9″.
(8) 75°, 60°, 45°. (9) 135°, 30°, 15°.

## LXVII. Page 211.

(1) 313·46 yds. (2) 28·87 inches, 31·43 inches.
(3) 1192·55 yds. (4) 22·415 ft.
(5) 24·995 = 25 ft. nearly, 17·559 ft., 65° 59′ 42″.

## LXVIII. Pages 213, 214.

(1) 108° 36′ 30″, 31° 23′ 30″. (2) 93° 11′ 49″, 36° 48′ 11′.
(3) 57″ 27′ 25·4″, 62° 32′ 34·6″. (4) 64° 26′ 47″, 37°, 7′ 13″.
(5) 72° 12′ 59″. (6) 20·5 chains. (7) 122·7.
(8) 74° 13′ 50″, 35° 16′ 10″.

## LXIX. Pages 218, 219.

(1) $A=61^\circ 18' 21''$, $C=88^\circ 41' 39''$; or $A=128^\circ 41' 39''$, $C=11^\circ 18' 21''$.
(2) $B=70^\circ 0' 56''$, $C=59^\circ 59' 4''$; or, $B=109^\circ 59' 4''$, $C=20^\circ 0' 56''$.
(3) $B=38^\circ 38' 24''$, $C=91^\circ 21' 36''$, $c=155·3$. (4) 61° 16′ 10″.
(5) $A=72^\circ 4' 48''$, $B=41^\circ 56' 12''$; or, $A=107^\circ 55' 12''$,
$B=6^\circ 5' 48''$, $b=17·56$. (6) $\beta$ is ambiguous; 60·3893 ft.

## LXX. Page 220.

The angles are given correct to the nearest second.

(1) $28^\circ 35' 39''$. (2) $104^\circ 44' 39''$. (3) $32^\circ 20' 48''$.
(4) $43^\circ 40'$. (5) $128^\circ 28' 13''$. (6) 106531 ft.
(7) 3437·6 yds. (8) 1728·2 chains. (9) 25376 yards.
(10) $A=66^\circ 27' 48'', B=12^\circ 55' 12''$. (11) $A=92^\circ 12' 58'', B=35^\circ 37' 7''$.
(12) $B=29^\circ 1' 40'', C=74^\circ 55' 50''$. (13) $B=70^\circ 35' 24''$; or, $109^\circ 24' 36''$.
(14) $B=51^\circ 56' 17''$; or, $128^\circ 3' 43''$. (15) $B=62^\circ 6' 10''$; or, $117^\circ 53' 50''$.
(16) Very nearly $90^\circ$. (17) 1319·6 yds.

## LXX b. P. 220 (i), (ii).

(1) $\cos A=\frac{1}{2}$, $\cos\frac{1}{2}A=\frac{1}{2}\sqrt{3}$. (2) $45^\circ, 60^\circ, 75^\circ$.
(3) $135^\circ, 30^\circ, 15^\circ$. (4) 3. (5) 14. (6) $1+\sqrt{3}$. (7) $120^\circ$.
(8) $120^\circ$. (9) $120^\circ$. (10) $90^\circ, 36^\circ 52'$. (11) $130^\circ 27'$. (12) $125^\circ 6'$.
(13) $120^\circ$. (14) $A=54^\circ$ or $126^\circ$, $B=108^\circ$ or $36^\circ$. (15) $a=1$.
(16) $C=30^\circ$, $a=\sqrt{3}+1$, $b=2$. (17) $A=75^\circ$, $a=b=2\sqrt{3}+1$.
(18) $C=60^\circ$ or $120^\circ$. (19) $100\sqrt{3}$. (20) no.
(22) $A=105^\circ$, $C=60^\circ$, $B=15^\circ$. (23) $\frac{1}{2}\sqrt{3}(\sqrt{5}+1)$.
(24) $A=90^\circ$ or $60^\circ$, $C=75^\circ$ or $105^\circ$, $a=2\sqrt{2}$ or $\sqrt{6}$. (25) $30^\circ$ or $150^\circ$. (26) $A=45^\circ$ or $135^\circ$, $B=30^\circ$ or $120^\circ$, $b=2\sqrt{2}(1+\sqrt{3})$ or $2\sqrt{6}(1+\sqrt{3})$. (27) $60^\circ, 75^\circ$, 6 yds. (28) It is impossible.
(30) $15 : 8\sqrt{3} : 4\sqrt{3}+9$.

## LXXII. Page 229.

(15) To find the point $E$ in an unlimited straight line $CE$ at which a finite straight line $AB$ subtends the greatest angle, a circle must be described passing through $A$ and $B$, and touching the line $CE$ in the point $E$. In (15) the centre of the circle lies vertically above $E$, and in the horizontal line through the middle point of $AB$.

## LXXIII. Page 239.

(1) (i) 10 sq. ft. (ii) 43·3 sq. in. (iii) 148·13 sq. yds.
(iv) 84 sq. chains = 8·4 acres. (v) 100 sq. ft.
(vi) 151872 sq. yds.
(2) 4, $10\frac{1}{2}$, 12, 14 ft. (4) $8\frac{1}{8}$ ft.

## LXXVI a. Pages 253—268.

(1) $\sin A=\frac{3}{5}$, $\cos A=\frac{4}{5}$. (3) $A=n\times180^0$; or, $360^0\pm60^0$.

(6) 4227·47 feet.

(9) $30^0$, $60^0$, $90^0$, $120^0$, etc. have for sine $\frac{1}{2}$, $\sqrt{\frac{3}{2}}$, 1, $\sqrt{\frac{3}{2}}$, $\frac{1}{2}$, 0, $-\frac{1}{2}$, $-\sqrt{\frac{3}{2}}$ $-1$, $-\sqrt{\frac{3}{2}}$, $-\frac{1}{2}$ respectively.

(12) The other sides are 765·4321 ft.; 1006·6 ft.

(15) $30^0$, $60^0$, $90^0$, etc. have for tan $\frac{1}{3}\sqrt{3}$, $\sqrt{3}$, $\infty$, $-\sqrt{3}$, $-\frac{1}{3}\sqrt{3}$, 0, $\frac{1}{3}\sqrt{3}$, $\infty$, $-\sqrt{3}$, $-\frac{1}{3}\sqrt{3}$ respectively.

(17) $\frac{168}{193}, \frac{168}{195}, \frac{32592}{193\times195}$. (19) $\sec A=\frac{3}{4}\sqrt{2}$.

(21) $\theta=\frac{1}{4}(2n+1)\pi$. (22) $\cos 5a=16\cos^5 a-20\cos^3 a+5\cos a$.

(23) Both signs negative. (24) $\frac{313}{338}$

(25) $\frac{11}{630}A$ radians; $5\cdot72956^0$.

(26) sine, $\frac{4}{5}$; tan, $\frac{4}{3}$; cot, $\frac{3}{4}$; cosec, $\frac{5}{4}$; sec, $\frac{5}{3}$.

(31) $\frac{630}{11}\theta$ deg.; $19\cdot09854^0$. (36) $C=18^0$, $a=c=\frac{2}{\sqrt{(10+2\sqrt{5})}}$.

(37) $-1320^0$. (38) $-\frac{2}{3}\sqrt{2}$. (39) $(2n+1)\pi$; or, $2n\pi\pm\frac{1}{3}\pi$.

(41) 1 foot, $120^0$, $30^0$; or 2 feet, $60^0$, $90^0$. (42) $104^0\,28'\,39''$.

(43) $-630^0$. (44) $-\frac{1}{5}\sqrt{5}$. (46) $n\pi$; and $2n\pi\pm\frac{2}{3}\pi$.

(48) $\frac{1}{2}\{\sqrt{6}\pm\sqrt{2}\}$ and $15^0$, $135^0$; or, $105^0$, $45^0$.

(49) $9^0$; $286^0\,.\,28'\,.\,41\cdot16''$; $\frac{7}{8}$.

(52) (i) $n\pi\pm\frac{1}{3}\pi$. (ii) $\frac{1}{2}n\pi\pm\frac{1}{8}\pi$, or $n\pi+(-1)^n\frac{1}{4}\pi$.

(53) $37\frac{1}{2}$ sq. ft. (54) $40^0\,.\,29'\,.\,19\cdot85''$.

(55) $\frac{1}{1080}\pi$; $\frac{1}{10}\pi$; $\frac{5}{4}$. (56) $\tan a=4\sqrt{3}$, $\operatorname{cosec} a=\frac{7}{12}\sqrt{3}$.

(58) (i) $n\pi\pm\frac{1}{4}\pi$. (ii) $\frac{1}{2}n\pi\pm\frac{1}{8}\pi$; or, $2n\pi\pm\frac{1}{4}\pi$.

(59) $2\sqrt{14}$ sq. ft. (60) $38^0\,.\,25'\,.\,32\cdot725''$.

(61) 1·2 radians = $76\cdot39416^g$.

(64) The proper formula is $+\sqrt{(1+\sin A)}+\sqrt{(1-\sin A)}$.

(66) 192 ft., 185 ft. and 9234 sq. ft.

(67) 2·3 radians = $131\cdot779926^0$.

(69) The proper formula is $-\sqrt{(1+\sin A)}-\sqrt{(1-\sin A)}$.

(72) $78^0\,10'$, $70^0\,30'$, 9234 sq. ft.

(73) $\frac{4}{35}\pi$; $\frac{9}{35}\pi$; $\frac{14}{35}\pi$; $\frac{19}{35}\pi$; $\frac{24}{35}\pi$.

(76) $(2n\pm\frac{1}{2})\pi$, or $(2n\pm\frac{1}{3})\pi$. (78) $135^0$, $15^0$; or $45^0$, $105^0$.

(79) $\frac{2}{21}\pi$; $\frac{4}{21}\pi$; $\frac{6}{21}\pi$; $\frac{8}{21}\pi$; $\frac{10}{21}\pi$; $\frac{12}{21}\pi$. (81) $(2n\pm\frac{1}{3})\pi$.

(84) 7 ft.; $\sqrt{19}$ ft.; $\frac{15}{2}\sqrt{3}$ sq. ft.

(88) 1035·43 ft.; 765·4321 ft.; $66^0$. (89) 6·981 feet.

(90) $\frac{1}{2}$; $-\frac{1}{2}$. (91) $n\pi+(-1)^n\frac{5}{4}\pi$. (94) 321·0793.

(95) 13·751 ft. (96) $\frac{12}{13}$. (97) $2n\pi\pm\frac{2}{3}\pi$.

(99) $2\cos\frac{1}{2}A = +\sqrt{(1+\sin A)} - \sqrt{(1-\sin A)}$.

(102) $x = \frac{2}{5}r\pi \pm \frac{2}{15}\pi \pm \frac{1}{10}\pi$, $y = \frac{2}{5}r\pi \pm \frac{1}{5}\pi \pm \frac{1}{15}\pi$.

(104) $x = \frac{1}{2}$. (105) 11·7157 miles. (107) $n\pi + \frac{1}{3}\pi$; or, $n\pi + \frac{1}{4}\pi$.

(110) 25·7834 yds. (115) 1·219714 miles and 1 mile.

(116) $\sin = \frac{63}{65}$, $\cos = -\frac{16}{65}$, $\tan = -\frac{63}{16}$.

(127) $4\theta \pm \theta = 2n\pi \pm \frac{3}{4}\pi$; or, $\frac{1}{2}(2n+1)\pi$.

(134) $5 + \frac{1}{2}\log 2 + \frac{1}{2}L\sec A = 1 - \log 2 + \frac{2}{3}L\cos B + \frac{1}{3}L$ cosec.

(139) $m\pi$; or, $n\theta = r\pi + (-1)^r(\frac{1}{2}\pi - \theta)$.

(144) $\log 5 = 2a - c$, $\log 7 = c - a$, $\log 13 = b - 4a + 2c$.

(145) In $\frac{1}{4}(2\sqrt{2} \pm 1)$ hours. (151) $60^0$, $45^0$, $135^0$, $120^0$.

(156) 1. (157) $30^0$ and $40^0$.

(161) $\theta = n\pi$ or $2\theta = n\pi + \frac{1}{4}\pi$. (164) $\cos\theta = \frac{1}{4}(1 \pm \sqrt{5})$.

(168) $\sin 2\theta = 0$ or $\cos 2\theta = \frac{1}{3}$ or $-\frac{1}{2}$. (178) $\theta$.

(182) $\dfrac{600}{13^2\sqrt{26}}$ and $\dfrac{5}{12}$. (191) $m^2 + p^2 = n^2 + q^2$.

(192) $$-4\cos\frac{B+C-A}{2}\cos\frac{C+A-B}{2}\cos\frac{A+B-C}{2}\cos\frac{A+B+C}{2}.$$

(196) $\frac{1}{2}\pi - \theta = n\pi + (-1)^n(\frac{1}{2}\pi - a)$.

(198) $$-4\cos\frac{90^0-B-C+A}{2}\cos\frac{90^0-C-A+B}{2}$$
$$\cos\frac{90^0-A-B+C}{2}\sin\frac{A+B+C-90^0}{2}.$$

## LXXVI b. Pages 269—275.

(10) $-(x+y+z)(y+z-x)(z+x-y)(x+y-z)$.

(11) (i) $\frac{1}{18}\{6n\pi - 2\pi + (-1)^n\pi\}$. (ii) $\dfrac{4r\pi+\pi}{2(m \pm n)}$.

(iii) $x = \dfrac{a-\beta}{2} + \cot^{-1}\left\{\tan\dfrac{a+\beta}{2}\cdot\dfrac{m\sin\beta + n\sin a}{n\sin a - m\sin\beta}\right\}$.

(iv) $\dfrac{2r\pi+\pi}{2(m+n)}$. (v) $\sin 2\theta(3\cos 2\theta - 1)(2\cos 2\theta + 1) = 0$.

(vi) $\sin 4\theta . \cos\dfrac{5\theta}{2} . \sin\dfrac{3\theta}{2} = 0$. (vii) $\sin 8\theta . \sin 4\theta = 0$.

(21) 223·17.

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